

RIEMANN-ROCH FOR EQUIVARIANT K -THEORY

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ABSTRACT. The goal of this paper is to prove the equivariant version of Bloch's Riemann-Roch isomorphism between the higher algebraic K -theory and the higher Chow groups of smooth varieties. We show that for a linear algebraic group G acting on a smooth variety X , although there is no Chern character map from the equivariant K -groups to equivariant higher Chow groups, there is indeed such a map $K_i^G(X) \otimes_{R(G)} \widehat{R(G)} \xrightarrow{ch} CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)}$ with rational coefficients, which is an isomorphism. This implies the Riemann-Roch isomorphism $\widehat{K_i^G(X)} \xrightarrow{\widehat{\tau_X^G}} CH_G^*(X, i)$. The case $i = 0$ provides a stronger form of the Riemann-Roch theorem of Edidin and Graham (cf. [6]).

1. INTRODUCTION

A *variety* in this paper will mean a reduced, connected and separated scheme of finite type with an ample line bundle over a field k . This base field k will be fixed throughout this paper. Let G be a linear algebraic group over k acting on such a variety X . Recall that this action on X is said to be *linear* if X admits a G -equivariant ample line bundle, a condition which is always satisfied if X is normal (cf. [26, Theorem 2.5] for G connected and [27, 5.7] for G general). All G -actions in this paper will be assumed to be linear. For $i \geq 0$, let $K_i^G(X)$ (resp. $G_i^G(X)$) denote the i th homotopy group of the K -theory spectrum of G -equivariant vector bundles (resp. coherent sheaves) on X . The G -equivariant higher Chow groups $CH_G^*(X, i)$ of X were defined by Edidin and Graham (cf. [7], also see below for more detail) as the ordinary higher Chow groups of the quotient space $X \times^G U$, where U is an open subscheme of a representation of G on which the action of G is free, and its complement is of sufficiently high codimension.

It has been known for a long time that in the non-equivariant case, there is a Riemann-Roch isomorphism $G_0(X)_{\mathbb{Q}} \rightarrow CH^*(X, 0)_{\mathbb{Q}}$ (cf. [11]). It is an important question to know if there are functorial Chern character and Riemann-Roch maps from K -theory spaces to a given cohomology theory, and if these maps are isomorphisms with rational coefficients. It was proved by Bloch (cf. [2, Theorem 9.1]) that for a quasi-projective variety X , there are Riemann-Roch maps $G_i(X) \rightarrow CH^*(X, i)$ which are isomorphisms with rational coefficients. It then follows that if X is smooth, the Chern character maps $K_i(X)_{\mathbb{Q}} \rightarrow CH^*(X, i)_{\mathbb{Q}}$ defined by Gillet in [12] are also isomorphisms.

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In this paper, we address the question of extending this result to the equivariant setting. It is not difficult to see however that unlike in the non-equivariant case, one can not expect such an isomorphism between the equivariant K -groups and higher Chow groups without perturbing them in some way. For example, for a finite cyclic group G of order m , one knows that $K_0^G(k)_\mathbb{Q}$ is a \mathbb{Q} -vector space of rank m , while $CH_G^*(X, 0)_\mathbb{Q}$ is a \mathbb{Q} -vector space of rank 1. This makes the equivariant Riemann-Roch problem more subtle than the ordinary one.

For finite group actions, the structure of $G^G(X)$ was analysed in [37]. The case that G acts on a smooth variety X with finite stabilizers was treated in [32] and [33]. In this case, one shows that $G^G(X)_\mathbb{Q}$ splits as a product from which one can deduce the equivariant Riemann-Roch for $CH_G^*(X, 0)$.

The situation is much more subtle when the stabilizers are not finite. Edidin and Graham proved the following fact. Denote by $R(G)$ the ring of representations of G , tensored with \mathbb{Q} ; call $I_G \subset R(G)$ the ideal of virtual representations of rank zero.

Theorem 1.1 (Edidin-Graham, [6]). *Let X be a separated algebraic space, and let $\widehat{G_0^G(X)}$ denote the I_G -adic completion of the $R(G)$ -module $G_0^G(X)_\mathbb{Q}$. Then there is a Riemann-Roch map*

$$\widehat{\tau}_X^G : \widehat{G_0^G(X)} \rightarrow \prod_{j=0}^{\infty} CH_G^j(X, 0)_\mathbb{Q},$$

which is an isomorphism.

Now, one can ask, first, if the completions of Edidin-Graham at the G_0 -level is the minimal perturbation one requires to define and prove Riemann-Roch isomorphisms in the equivariant setting, and second, if such isomorphisms could be established also for the higher equivariant K -groups. Our goal in this paper is to answer these two questions, in the particular case that X is smooth. We show that the difference between the equivariant and the non-equivariant cases occurs already at the level of K_0 of the base field: once this difference is taken into account, one can define and prove the Riemann-Roch and Chern character isomorphisms for equivariant higher K -theory just like Bloch's theorem in the non-equivariant case.

We set up some notations before we state our main result. For a linear algebraic group G over k as above, let $R(G)$ denote the ring of virtual representations of G and let I_G be the ideal of the rank zero virtual representations, i.e., I_G is the kernel of the rank map $R(G) \rightarrow \mathbb{Z}$. It is easy to see that $R(G)$ is same as the Grothendieck group $K_0^G(k)$. Let $\widehat{R(G)}$ denote the completion of the ring $R(G)$ with respect to the ideal I_G . Let $S(G)$ denote the equivariant Chow ring $CH_G^*(k, 0) = \bigoplus_{j \geq 0} CH_G^j(k, 0)$ and let J_G denote the irrelevant ideal $\bigoplus_{j \geq 1} CH_G^j(k, 0)$. Let $\widehat{S(G)}$ denote the J_G -adic completion of $S(G)$. We shall call I_G and J_G as the augmentation ideals of the rings $R(G)$ and $S(G)$ respectively, conforming to the notations already in use in the literature. Let $\widehat{I_G}$ (resp. $\widehat{J_G}$) denote the extension of I_G (resp. J_G) in the completion $\widehat{R(G)}$ (resp. $\widehat{S(G)}$).

For a variety X with a G -action, put

$$CH_G^*(X, i) = \bigoplus_{j \geq 0} CH_G^j(X, i) \quad \text{for } i \geq 0.$$

It is known (cf. [7]) that the term on the right is an infinite sum in general. For any such variety X , $G_i^G(X)$ and $K_i^G(X)$ are $K_0^G(X)$ -modules and hence the ring homomorphism $R(G) \rightarrow K_0^G(X)$ makes $G_i^G(X)$ and $K_i^G(X)$ $R(G)$ -modules. Moreover, for any G -equivariant map $f : X \rightarrow Y$, the pull-back map $f^* : K_i^G(Y) \rightarrow K_i^G(X)$ is $R(G)$ -linear, and the projection formula (cf. [15, Ex. II.8.3]) implies that so is the push-forward map if f is proper. At the level of equivariant higher Chow groups, the smoothness of the classifying stack BG implies that there is an action of $CH_G^*(k, 0)$ on $CH_G^*(X, i)$ (cf. [2, Proposition 5.5]) and this makes the latter an $S(G)$ -module. As in the case of K -groups, any G -equivariant map $f : X \rightarrow Y$ induces an $S(G)$ -linear pull-back map on the equivariant higher Chow groups if Y is smooth, and an $S(G)$ -linear push-forward map if f is proper (cf. [2, 5.8]).

For a G -variety X , let $\widehat{G_i^G(X)}$ denote the I_G -adic completion of the $R(G)$ -module $G_i^G(X)$, and let $\widehat{CH_G^*(X, i)}$ denote the J_G -adic completion of the $S(G)$ -module $CH_G^*(X, i)$. Finally, we recall that if X is smooth, then there is a Poincaré duality isomorphism of spectra $K^G(X) \xrightarrow{\cong} G^G(X)$ (cf. [27, Theorem 1.8]). All the K -theory and Chow groups in this paper (except in Section 2) will be tensored with \mathbb{Q} . We now state our main result.

Theorem 1.2. *Let X be a smooth variety with a G -action. Then for any $i \geq 0$, there are Chern character maps*

$$(1.1) \quad \widetilde{ch}_X^G : K_i^G(X) \otimes_{R(G)} \widehat{R(G)} \longrightarrow CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)}$$

$$(1.2) \quad \widehat{ch}_X^G : \widehat{K_i^G(X)} \longrightarrow \widehat{CH_G^*(X, i)}$$

and a commutative diagram

$$(1.3) \quad \begin{array}{ccc} K_i^G(X) \otimes_{R(G)} \widehat{R(G)} & \xrightarrow{\widetilde{ch}_X^G} & CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)} \\ u_X^G \downarrow & & \downarrow \overline{u}_X^G \\ \widehat{K_i^G(X)} & \xrightarrow{\widehat{ch}_X^G} & \widehat{CH_G^*(X, i)} \end{array}$$

such that the horizontal maps are isomorphisms and the vertical maps are injective. Moreover, these Chern character maps commute with the pull-back maps on K -groups and higher Chow groups of smooth G -varieties, and with products.

Corollary 1.3. *Let X be a smooth variety with a G -action. Then for every $i \geq 0$, there is a Riemann-Roch isomorphism*

$$\widehat{\tau}_X^G : \widehat{K_i^G(X)} \xrightarrow{\cong} \widehat{CH_G^*(X, i)}.$$

This result is the correct generalization of the Riemann-Roch theorem of Edidin and Graham [6] to the higher K -theory. In fact for $i = 0$, one knows (cf. [13, Proposition 2.1], [4, Corollary 2.3]) that the J_G -adic and the graded filtrations induce the same topology on $CH_G^*(X, 0)$. Hence the natural map $\widehat{CH_G^*(X, i)} \rightarrow \prod_{j=0}^{\infty} CH_G^j(X, 0)$

is an isomorphism. In particular, the main result of Edidin and Graham [6] is a special case of Corollary 1.3.

For actions with finite stabilizers, the above results can be further refined to give the following stronger form which is strikingly similar to the Bloch's non-equivariant Riemann-Roch theorem. For $i = 0$, the Riemann-Roch in this form was conjectured by Vistoli (cf. [36]) and proved (without the surjectivity assertion) by Edidin and Graham (cf. [6, Corollary 5.2]).

Theorem 1.4. *Let G act on a (possibly singular) variety X with finite stabilizers. Assume that X is either smooth or, the stack $[X/G]$ has a coarse moduli scheme. Then for $i \geq 0$, the Riemann-Roch map of Theorem 9.1 induces a map*

$$G_i^G(X) \xrightarrow{\tau_X^G} CH_G^*(X, i),$$

which is surjective, and $\alpha \in \text{Ker}(\tau_X^G)$ if and only if there exists a virtual representation $\epsilon \in R(G)$ of non-zero rank such that $\epsilon\alpha = 0$.

We make a few remarks on the above results. First of all, our results seem to be best possible form of equivariant Riemann-Roch Theorem one could possibly hope for. This is because it is unavoidable to tensor the left and the right sides of 1.1 with $\widehat{R(G)}$ and $\widehat{S(G)}$ respectively, as can be seen even at the level of finite group actions.

The ring $R(G)_{\mathbb{Q}}$ has in general infinitely many maximal ideals, besides the augmentation ideal I_G . For any maximal ideal \mathfrak{m} , let $\widehat{R(G)}_{\mathfrak{m}}$ denote the \mathfrak{m} -adic completion of $R(G)_{\mathbb{Q}}$, and let $\widehat{G_i^G(X)}_{\mathfrak{m}}$ denote the tensor product $G_i^G(X) \otimes_{R(G)} \widehat{R(G)}_{\mathfrak{m}}$. Then the results of this paper give a description of the group $\widehat{G_i^G(X)}_{I_G}$ in terms of equivariant higher Chow groups. One would like to give a similar description of $\widehat{G_i^G(X)}_{\mathfrak{m}}$, for any given maximal ideal \mathfrak{m} . This will be the subject of a forthcoming sequel to this paper.

We end this section with a brief description of the contents of the various sections of this paper. We review the definitions and various properties of the equivariant higher Chow groups in the next section. The main result here is a self intersection formula for the higher Chow groups. This formula is crucial for the decomposition theorem of equivariant higher Chow groups of varieties with an action of a diagonalizable group. In Section 3, we prove various reduction techniques for describing the equivariant higher Chow groups for action of arbitrary groups in terms of the higher Chow groups for action of tori. We also prove a decomposition theorem for the equivariant higher Chow groups of a G -variety X when certain subgroup of G acts trivially on X . Section 4 is devoted to the description and comparison of various completions of the $S(G)$ -modules $CH_G^*(X, \cdot)$ for a G -variety X . In Section 5,

we study the notion of cohomological rigidity, which is then used to construct various specialization maps between the equivariant higher Chow groups, analogous to the specialization maps in equivariant K -theory in [34]. In the following section, we prove our main decomposition theorem (Theorem 6.3) for the equivariant higher Chow groups for action of diagonalizable groups. This is one of the crucial steps in proving Theorem 1.2 for diagonalizable groups. In Section 7, we construct our main objects of study, the equivariant Chern character and Riemann-Roch maps. We also prove various properties of these maps which are useful in proving our main results. Our approach to the proof of our main result is to eventually reduce it to the case when the underlying group acts either with finite stabilizers or with a constant dimension of stabilizers. Sections 8 and 9 are devoted to proving our results in these cases for the action of diagonalizable groups. In Section 10, we prove our main result for actions of diagonalizable groups. Section 11 is devoted to proving some results for relating the equivariant K -groups and higher Chow groups for actions of any linear algebraic groups with these groups for actions of diagonalizable groups. We finally prove the above stated results in the last section of the paper. We also give an application of the above Riemann-Roch theorems to the equivariant K -theory.

2. EQUIVARIANT HIGHER CHOW GROUPS

Unlike the case of the rest of this paper, all the groups in this section will be considered with integral coefficients. We begin with a brief review of the equivariant higher Chow groups from [7] and their main properties, especially those which will be used repeatedly in this paper. Our main result in this section is the self intersection formula for the higher Chow groups of smooth varieties. The analogous formula for the higher K -theory was proved by Thomason (cf. [28, Theorem 3.1]). Surprisingly, this formula for the higher Chow groups has still been unknown. The equivariant version of this formula will be crucial in proving our decomposition Theorem 6.3 for the equivariant higher Chow groups of smooth varieties with an action of a diagonalizable group.

Let G be a linear algebraic group and let X be an equidimensional variety over k with a G -action. All representations of G in this paper will be finite dimensional. The definition of equivariant higher Chow groups of X needs one to consider certain kind of mixed spaces which in general may not be a scheme even if the original space is a scheme. The following well known (cf. [7, Proposition 23]) lemma shows that this problem does not occur in our context and all the mixed spaces in this paper are schemes with ample line bundles.

Lemma 2.1. *Let H be a linear algebraic group acting freely and linearly on a k -variety U such that the quotient U/H exists as a quasi-projective variety. Let X be a k -variety with a linear action of H . Then the mixed quotient $X \times^H U$ exists for the diagonal action of H on $X \times U$ and is quasi-projective. Moreover, this quotient is smooth if both U and X are so. In particular, if H is a closed subgroup of a linear algebraic group G and X is a k -variety with a linear action of H , then the quotient $G \times^H X$ is a quasi-projective scheme.*

Proof. It is already shown in [7, Proposition 23] using [9, Proposition 7.1] that the quotient $X \times^H U$ is a scheme. Moreover, as U/H is quasi-projective, [9, Proposition 7.1] in fact shows that $X \times^H U$ is also quasi-projective. The similar conclusion about $G \times^H X$ follows from the first case by taking $U = G$ and by observing that G/H is a smooth quasi-projective scheme (cf. [3, Theorem 6.8]). The assertion about the smoothness is clear since $X \times U \rightarrow X \times^H U$ is a principal H -bundle. \square

For any integer $j \geq 0$, let V be a representation of G and let U be a G -invariant open subset of V such that the codimension of the complement $V - U$ in V is larger than j , and G acts freely on U such that the quotient U/G is a quasi-projective scheme. Such a pair (V, U) will be called a *good pair* for the G -action corresponding to j . It is easy to see that a good pair always exists (cf. [7, Lemma 9]). Let X_G denote the quotient $X \times^G U$ of the product $X \times U$ by the diagonal action of G which is free. We define the equivariant higher Chow group $CH_G^j(X, i)$ as the homology group $H_i(\mathcal{Z}(X_G, \cdot))$, where $\mathcal{Z}(X_G, \cdot)$ is the Bloch's cycle complex of the variety X_G . It is known (*loc. cit.*) that this definition of $CH_G^j(X, i)$ is independent of the choice of a good pair (V, U) for the G -action. One should also observe that $CH_G^j(X, i)$ may be non-zero for infinitely many values of j , a crucial change from the non-equivariant higher Chow groups. The following result summarizes most of the essential properties of the equivariant higher Chow groups that will be used in this paper. Let \mathcal{V}_G denote the category of G -varieties with G -equivariant maps and let \mathcal{V}_G^S denote the full subcategory of smooth G -varieties.

Proposition 2.2. *The equivariant higher Chow groups as defined above satisfy the following properties.*

- (i) *Functoriality : Covariance for proper maps and contravariance for flat maps. Moreover, if $f : X \rightarrow Y$ is a morphism in \mathcal{V}_G with Y in \mathcal{V}_G^S , then there is a pull-back map $f^* : CH_G^*(Y, i) \rightarrow CH_G^*(X, i)$.*
- (ii) *Homotopy : If $f : X \rightarrow Y$ is an equivariant vector bundle, then $f^* : CH_G^*(Y, i) \xrightarrow{\cong} CH_G^*(X, i)$.*
- (iii) *Localization : If $Y \subset X$ is of pure codimension d with complement U , then there is a long exact localization sequence*

$$\cdots \rightarrow CH_G^{*-d}(Y, i) \rightarrow CH_G^*(X, i) \rightarrow CH_G^*(U, i) \rightarrow CH_G^{*-d}(Y, i-1) \rightarrow \cdots$$

- (iv) *Exterior product : There is a natural product map*

$$CH_G^j(X, i) \otimes CH_G^{j'}(Y, i') \rightarrow CH_G^{j+j'}(X \times Y, i+i').$$

Moreover, if $f : X \rightarrow Y$ is such that $Y \in \mathcal{V}_G^S$, then there is a pull-back via the graph map $\Gamma_f : X \rightarrow X \times Y$, which makes $CH_G^(Y, \cdot)$ a bigraded ring and $CH_G^*(X, \cdot)$ a module over this ring.*

- (v) *Chern classes : For any G -equivariant vector bundle of rank r , there are equivariant Chern classes $c_l^G(E) : CH_G^j(X, i) \rightarrow CH_G^{j+l}(X, i)$ for $1 \leq l \leq r$, having the same functoriality properties as in the non-equivariant case.*

- (vi) *Projection formula* : For a proper map $f : X \rightarrow Y$ in \mathcal{V}_G with $Y \in \mathcal{V}_G^S$, one has for $x \in CH_G^*(X, \cdot), y \in CH_G^*(Y, \cdot)$, $f_*(x \cdot f^*(y)) = f_*(x) \cdot y$.
- (vii) *Free action* : If G acts freely on X with quotient Y , then there is a natural isomorphism $CH_G^*(X, i) \xrightarrow{\cong} CH^*(Y, i)$.

Proof. Since the equivariant higher Chow groups of X are defined in terms of the higher Chow groups of X_G , the proposition (except possibly the last property) can be easily deduced from the similar results for the non-equivariant higher Chow groups as in [2] and the techniques of *loc. cit.*. We therefore skip the proof. For the last property, fix $j \geq 0$ and choose a good pair (V, U) for the G -action corresponding to j . Since G acts freely on X , it acts likewise also on $X \times V$ with quotient, say X_V . Then X_G is an open subset of X_V and $X_V \rightarrow Y$ is a locally trivial V -fibration, which implies that the map $CH^j(Y, i) \rightarrow CH^j(X_V, i)$ is an isomorphism by the homotopy invariance. On the other hand, the pull-back map $CH^j(X_V, i) \rightarrow CH^j(X_G, i) = CH_G^j(X, i)$ is an isomorphism by the property (iii) as j is sufficiently large. \square

If X is not equidimensional, then one defines the equivariant higher Chow groups $CH_j^G(X, i)$ as $CH_{j+l-g}(X_G, i)$, where X_G is formed from an l -dimensional representation V such that $V - U$ has sufficiently high codimension and g is the dimension of G . The groups $CH_j^G(X, i)$ enjoy many of the properties stated above and in particular, one has the localization sequence as above even if the closed subscheme $Y \subset X$ is not equidimensional (*cf.* [loc. cit., Proposition 5]). It is easy to see that $CH_j^G(X, i) \cong CH_G^{d-j}(X, i)$ if X is equidimensional of dimension d .

We next recall that the Chern classes $c_i^G(E)$ of an equivariant vector bundle E , as described in Proposition 2.2 above, live in the operational Chow groups $A^l(X_G)$. If X is in \mathcal{V}_G^S however, this operational Chow group is isomorphic to the equivariant Chow group $CH_G^l(X, 0)$ and the action of $c_i^G(E)$ on $CH_G^*(X, \cdot)$ then coincides with the intersection product in the ring $CH_G^*(X, \cdot)$.

Finally, we recall from *loc. cit.* that if $H \subset G$ is a closed subgroup and if (V, U) is a good pair, then for $X \in \mathcal{V}_G$, the natural map of quotients $X \times^H U \rightarrow X \times^G U$ is a G/H -principal bundle and hence there is a natural restriction map

$$(2.1) \quad r_H^G : CH_G^*(X, \cdot) \rightarrow CH_H^*(X, \cdot).$$

Taking $H = \{1\}$, and using the homotopy invariance and the localization sequence, one gets a natural map

$$(2.2) \quad r_X^G : CH_G^*(X, \cdot) \rightarrow CH^*(X, \cdot).$$

Moreover, as r_X^G is the pull-back under a flat map, it commutes (*cf.* Proposition 2.2) with the pull-back for any flat map, and with the push-forward for any proper map in \mathcal{V}_G . We remark here that although the definition of r_H^G uses a good pair (V, U) for any given $j \geq 0$, it is easy to check from the homotopy invariance that r_H^G is independent of the choice of the good pair (V, U) .

As mentioned before, our main goal in this section is to prove a self-intersection formula for the ordinary and equivariant higher Chow groups. Our main technical

tool to prove this is the *deformation to the normal cone* method. Since this technique will be used several times in this paper, we briefly recall the construction from [11, Chapter 5] for our as well as reader's convenience. Let X be a smooth variety over k and let $Y \xrightarrow{f} X$ be a smooth closed subscheme of codimension $d \geq 1$. Let \widetilde{M} be the blow-up of $X \times \mathbb{P}^1$ along $Y \times \infty$. Then $Bl_Y(X)$ is a closed subscheme of \widetilde{M} and one denotes its complement by M . There is a natural map $\pi : M \rightarrow \mathbb{P}^1$ such that $\pi^{-1}(\mathbb{A}^1) \cong X \times \mathbb{A}^1$ with π the projection map and $\pi^{-1}(\infty) \cong X'$, where X' is the total space of the normal bundle $N_{Y/X}$ of Y in X . One also gets the following diagram, where all the squares and the triangles commute.

$$(2.3) \quad \begin{array}{ccccc} Y & \xleftarrow{p_Y} & Y \times \mathbb{P}^1 & \xleftarrow{i_\infty} & Y \\ & \searrow u' & \nearrow j' & & \downarrow f' \\ & & Y \times \mathbb{A}^1 & & \\ & \downarrow f & \downarrow F & & \\ X & \xrightarrow{F'} & M & \xleftarrow{i} & X' \\ & \searrow u & \nearrow j & & \\ & & X \times \mathbb{A}^1 & & \end{array}$$

In this diagram, all the vertical arrows are the closed embeddings, i_0 and i_∞ are the obvious inclusions of Y in $Y \times \mathbb{P}^1$ along the specified points, i and j are inclusions of the inverse images of ∞ and \mathbb{A}^1 respectively under the map π , u and f' are zero section embeddings and p_Y is the projection map. In particular, one has $p_Y \circ i_0 = p_Y \circ i_\infty = id_Y$.

We also make the observation here that in case X is a G -variety and Y is G -invariant, then by letting G act trivially on \mathbb{P}^1 and diagonally on $X \times \mathbb{P}^1$, one gets a natural action of G on M , and all the spaces in the above diagram become G -spaces and all the morphisms become G -equivariant. This observation will be used later on in this paper.

We shall need the following result about the higher Chow groups which is an easy consequence of Bloch's moving lemma.

Lemma 2.3. *Let*

$$\begin{array}{ccc} W & \xrightarrow{i'} & Y \\ j' \downarrow & & \downarrow j \\ Z & \xrightarrow{i} & X \end{array}$$

be a fiber diagram of closed immersions such that X and Y are smooth. Then one has $i^ \circ j_* = j'_* \circ i'^* : CH^*(Y, \cdot) \rightarrow CH^*(Z, \cdot)$.*

Proof. Since X and Y are smooth, we can assume them to be equidimensional.

Let $\mathcal{Z}_{ZW}^p(Y, \cdot) \xrightarrow{i_Y} \mathcal{Z}^p(Y, \cdot)$ be the subcomplex which is generated by cycles on $Y \times \Delta^1$ which intersect all faces of $Z \times \Delta^1$ and $W \times \Delta^1$ properly. Similarly, let

$\mathcal{Z}_Z^p(X, \cdot) \xrightarrow{i_X} \mathcal{Z}^p(X, \cdot)$ be the subcomplex generated by cycles on $X \times \Delta^1$ which intersect all faces of $Z \times \Delta^1$ properly. Then i_X and i_Y are quasi-isomorphisms by the moving lemma (cf. [21, Theorem 1.10]). However, if $V \in \mathcal{Z}_{ZW}^p(Y, \cdot)$ is an irreducible cycle in $Y \times \Delta^n$, then the conclusion of the lemma is checked easily. \square

Lemma 2.4. *Consider the diagram 2.3 and let $y \in CH^*(Y, m)$. Then there exists $z \in CH^*(M, m)$ such that $f_*(y) = h^*(z)$ and $f'_*(y) = i^*(z)$.*

Proof. Put $\tilde{y} = p_Y^*(y)$ and $z = F_*(\tilde{y})$. Then

$$\begin{aligned} f_*(y) &= f_*((p_Y \circ i_0)^*(x)) \\ &= f_* \circ i_0^* \circ p_Y^*(y) = f_* \circ i_0^*(\tilde{y}) \\ &= f_* \circ u'^* \circ j'^*(\tilde{y}) \\ &= u^* \circ F'_*(j'^*(\tilde{y})) && \text{(by Lemma 2.3)} \\ &= u^* \circ j^* \circ F_*(\tilde{y}) && \text{(since } j \text{ is an open immersion)} \\ &= h^* \circ F_*(\tilde{y}) = h^*(z). \end{aligned}$$

Similarly,

$$\begin{aligned} f'_*(y) &= f'_*((p_Y \circ i_\infty)^*(x)) \\ &= f'_* \circ i_\infty^* \circ p_Y^*(y) = f'_* \circ i_\infty^*(\tilde{y}) \\ &= i^* \circ F_*(\tilde{y}) && \text{(by Lemma 2.3)} \\ &= i^*(z). \end{aligned}$$

\square

Theorem 2.5 (Self-intersection Formula). *Let $Y \xrightarrow{f} X$ be a closed immersion of smooth varieties of codimension $d \geq 1$, and let $N_{Y/X}$ be the normal bundle of Y in X . Then one has for every $y \in CH^*(Y, \cdot)$, $f^* \circ f_*(y) = c_d(N_{Y/X}) \cdot y$.*

Proof. We first consider the case when $X \xrightarrow{p} Y$ is a vector bundle of rank d and f is the zero section embedding so that $p \circ f = id_Y$. In that case, we have

$$\begin{aligned} f^* \circ f_*(y) &= f^* \circ f_*(f^* \circ p^*(y)) \\ &= f^*(f_*(1) \cdot p^*(y)) && \text{(by Proposition 2.2 (vi))} \\ &= f^*(f_*(1)) \cdot (f^* \circ p^*(y)) \\ &= f^*(f_*(1)) \cdot y \\ &= c_d(N_{Y/X}) \cdot y, \end{aligned}$$

where the last equality follows from the self-intersection formula for Fulton's Chow groups (cf. [11, Corollary 6.3]). This proves the theorem in the case of zero section embedding.

Now let $Y \hookrightarrow X$ be as in the theorem. We consider the deformation to the normal cone diagram 2.3, and choose $z \in CH^*(M, \cdot)$ as in Lemma 2.4. Then we have

$$\begin{aligned} f^* \circ f_*(y) &= f^* \circ h^*(z) = i_0^* \circ F^*(z) \\ &= i_\infty^* \circ F^*(z) = f'^* \circ i^*(z) \\ &= c_d(N_{Y/X'}) \cdot y && \text{(by the case of vector bundle above)} \\ &= c_d(N_{Y/X}) \cdot y. \end{aligned}$$

This completes the proof of the theorem. \square

Corollary 2.6. *Let G be a linear algebraic group over k and let $Y \xrightarrow{f} X$ be a closed immersion of codimension $d \geq 1$ in \mathcal{V}_G^S . Then one has for every $y \in CH_G^*(Y, \cdot)$, $f^* \circ f_*(y) = c_d^G(N_{Y/X}) \cdot y$.*

Proof. Fix $i, j \geq 0$ and choose a good pair (V, U) for $n \gg j + d$. We can then identify $CH_G^p(X, i)$ with $CH^p(X_G, i)$ (and same for Y) for $p \leq n$. We can also identify $c_d^G(E)$ with $c_d(E_G)$ for any equivariant vector bundle E on Y (cf. [7, Section 2.4]). Now the proof of the corollary would follow straightaway from Theorem 2.5, once we show that $(N_{Y/X})_G$ is the normal bundle of Y_G in X_G . But this follows immediately from the elementary fact that if G acts freely on a smooth variety Z and W is a smooth closed and G -invariant subvariety of Z with normal bundle N , then G acts freely on N , and moreover, N/G is the normal bundle of W/G in Z/G . We leave the proof of this fact to the reader. \square

3. REDUCTION TECHNIQUES FOR EQUIVARIANT HIGHER CHOW GROUPS

One of the important tools in the equivariant geometry is the technique of reducing the study of varieties with action of an arbitrary linear algebraic groups to the reductive and then to the diagonalizable groups. In order to successfully apply this technique in practice, one often needs to know how certain invariants of varieties with an action of a group G are related to these invariants for the actions of various subgroups or quotients of G . In this section, we establish some basic results in this direction about the equivariant higher Chow groups. We recall here our convention that an abelian group A in the rest of this paper will actually mean the group $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Although many of the results that follow hold also with the integral coefficients, we shall not need them in that form.

Proposition 3.1 (Morita isomorphism). *Let H be a normal subgroup of a linear algebraic group G and let $F = G/H$. Let $f : X \rightarrow Y$ be a G -equivariant morphism of G -varieties which is an H -torsor for the restricted action. Then for every $i, j \geq 0$, the map f induces an isomorphism of the equivariant higher Chow groups*

$$CH_F^j(Y, i) \xrightarrow{f^*} CH_G^j(X, i).$$

Proof. We first observe from [25, Corollary 12.2.2] that F is also a linear algebraic group over the given ground field k . Now, since f is an H -torsor, it is clear that G acts on Y via F . Fix $j \geq 0$ and choose a good pair (V, U) for the F -action on Y corresponding to j . Then V is also a representation of G in which U is G -invariant. In particular, G acts on $X \times U$ via the diagonal action, which is easily seen to be free since H acts freely on X and F acts freely on U . By the same reason, we see that $X \times U \rightarrow Y \times U$ is G -equivariant which is a principal H -bundle. This in turn implies that the map $(X \times U)/G \rightarrow Y_F$ is an isomorphism and hence we get

$$(3.1) \quad CH_F^j(Y, i) \cong CH^j(Y_F, i) \xrightarrow{f^*} CH^j((X \times U)/G, i).$$

On the other hand, we have

$$CH_G^j(X, i) \cong CH_G^j(X \times V, i) \cong CH_G^j(X \times U, i) \cong CH^j((X \times U)/G, i),$$

where the first isomorphism is due to the homotopy invariance, the second follows from the localization property (cf. Proposition 2.2, (iii)) as j is sufficiently large, and the third isomorphism follows from Proposition 2.2, (vii). The proof of the proposition now follows by combining this with 3.1. \square

Corollary 3.2 (cf. [6]). *Let $H \subset G$ be a closed subgroup and let $X \in \mathcal{V}_H$. Then for any $i, j \geq 0$, there is a natural isomorphism*

$$(3.2) \quad CH_G^j \left(G \times^H X, i \right) \xrightarrow{\cong} CH_H^j (X, i).$$

Proof. Define an action of $H \times G$ on $G \times X$ by

$$(h, g) \cdot (g', x) = (gg'h^{-1}, hx),$$

and an action of $H \times G$ on X by $(h, g) \cdot x = hx$. Then the projection map $G \times X \xrightarrow{p} X$ is $(H \times G)$ -equivariant which is a G -torsor. Hence by Proposition 3.1, the natural map $CH_H^j (X, i) \xrightarrow{p^*} CH_{H \times G}^j (G \times X, i)$. On the other hand, the projection map $G \times X \rightarrow G \times^H X$ is $(H \times G)$ -equivariant which is an H -torsor. Hence we get an isomorphism $CH_G^j \left(G \times^H X, i \right) \xrightarrow{\cong} CH_{H \times G}^j (G \times X, i)$. The corollary follows by combining these two isomorphisms. \square

Theorem 3.3. *Let G be a connected and reductive group over k . Let B be a Borel subgroup of G containing a maximal torus T over k . Then the restriction maps*

$$(3.3) \quad CH_B^* (X, \cdot) \xrightarrow{r_T^B} CH_T^* (X, \cdot),$$

$$(3.4) \quad CH_G^* (X, \cdot) \xrightarrow{r_T^G} CH_T^* (X, \cdot)$$

are respectively isomorphism and split monomorphism. Moreover, this splitting is natural for morphisms in \mathcal{V}_G . In particular, if H is any closed subgroup of G , then then there is a split injective map

$$(3.5) \quad CH_H^* (X, \cdot) \xrightarrow{r_T^G} CH_T^* \left(G \times^H X, \cdot \right)$$

Proof. We first prove 3.3. By Corollary 3.2, we only need to show that

$$(3.6) \quad CH_B^* \left(B \times^T X, \cdot \right) \cong CH_T^* (X, \cdot).$$

By [5, XXII, 5.9.5], there exists a characteristic filtration $B^u = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n = \{1\}$ of the unipotent radical B^u of B such that U_{i-1}/U_i is a vector group, each U_i is normal in B and $TU_i = T \ltimes U_i$. Moreover, this filtration also implies that for each i , the natural map $B/BU_i \rightarrow B/TU_{i-1}$ is a torsor under the vector

bundle $U_{i-1}/U_i \times B/TU_{i-1}$ on B/TU_{i-1} . Hence, the homotopy invariance gives an isomorphism

$$CH_B^*(B/TU_{i-1} \times X, \cdot) \xrightarrow{\cong} CH_B^*(B/TU_i \times X, \cdot).$$

Composing these isomorphisms successively for $i = 1, \dots, n$, we get

$$CH_B^*(X, \cdot) \xrightarrow{\cong} CH_B^*(B/T \times X, \cdot).$$

The isomorphism of B -varieties $B \overset{T}{\times} X \cong B/T \times X$ (cf. Corollary 3.2) now proves 3.6 and hence 3.3.

To prove 3.4, we can apply 3.3 to reduce to showing that the map $CH_G^*(X, \cdot) \rightarrow CH_B^*(X, \cdot)$ is a naturally split monomorphism. By 3.2, there is an isomorphism $CH_B^*(X, \cdot) \cong CH_G^*\left(G \overset{B}{\times} X, \cdot\right)$. Moreover, there is an isomorphism of G -varieties

$G \overset{B}{\times} X \cong G/B \times X$. Thus it suffices to show for the flat and proper map $f : G/B \times X \rightarrow X$ that f^* is split by the map f_* . Using the projection formula of Proposition 2.2, it suffices to show that $f_*(1) = 1$. But this follows directly from [19, Theorem 2.1]. Finally, 3.5 follows from 3.4 and Corollary 3.2. \square

Proposition 3.4. *Let H be a possibly non-reductive group over k . Assume that H has a Levi decomposition $H = L \ltimes H^u$ such that H^u is split over k (e.g., when k is of characteristic zero). Then the restriction map*

$$(3.7) \quad CH_H^*(X, \cdot) \xrightarrow{r_L^H} CH_L^*(X, \cdot),$$

is an isomorphism.

Proof. Since the unipotent radical of H is split over k , the proof is exactly same as in the proof of 3.3, where we just have to replace B and T by H and L respectively. \square

Remark 3.5. We point out here that though we have assumed all abelian groups to be tensored with \mathbb{Q} in this and the latter sections, the readers can check from the proofs that the results of this section so far, remain true with the integral coefficients.

A consequence of Theorem 3.3 is that the equivariant higher Chow groups for the action of a connected reductive group G are subgroups of the equivariant higher Chow groups for the action of a maximal torus of G . In our next result, we prove a refinement of this by giving an explicit description of these subgroups. This is a generalization of the analogous result Proposition 6 of [7] to equivariant higher Chow groups. We begin with following result about the non-equivariant higher Chow groups.

Lemma 3.6. *If L is a linear variety over k and $X = Y \times L$, then the exterior product map*

$$CH^*(Y, \cdot) \otimes_{CH^*(k, \cdot)} CH^*(L, \cdot) \rightarrow CH^*(X, \cdot)$$

is an isomorphism of $CH^(k, \cdot)$ -modules.*

Proof. We prove by the induction on the dimension of L . If L is 0-dimensional, there is nothing to prove. So we assume that $\dim(L) \geq 1$ and that the lemma holds for all linear varieties of dimension less than the dimension of L . Put $CH^*(k) = \bigoplus_{i \geq 0} CH^*(k, i)$.

Since L is linear, there exists an open dense subset $U \subset L$ such that $U \cong \mathbb{A}^n$ for some n and $L' = L - U$ is a linear variety of dimension less than that of L . Moreover, the localization sequence

$$0 \rightarrow CH^*(L', \cdot) \rightarrow CH^*(L, \cdot) \rightarrow CH^*(U, \cdot) \rightarrow 0$$

is split exact (cf. [18]). In particular, this sequence remains exact after tensoring with $CH^*(Y, \cdot)$, and we get a diagram of localization exact sequences

$$\begin{array}{ccccccc} 0 \rightarrow CH^*(Y, \cdot) \otimes CH^*(L', \cdot) & \rightarrow & CH^*(Y, \cdot) \otimes CH^*(L, \cdot) & \rightarrow & CH^*(Y, \cdot) \otimes CH^*(U, \cdot) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & \longrightarrow & CH^*(Y \times L', \cdot) & \xrightarrow{i_*} & CH^*(Y \times L, \cdot) & \xrightarrow{j^*} & CH^*(Y \times U, \cdot) \longrightarrow, \end{array}$$

where the tensor product in the top row is over the ring $CH^*(k)$. The left vertical arrow is an isomorphism by induction, and the right vertical arrow is an isomorphism by the homotopy invariance. In particular, j^* is surjective in all indices. We conclude that i_* is injective in all indices and the middle vertical arrow is an isomorphism. \square

Recall that a connected and reductive group G over k is said to be *split*, if it contains a split maximal torus T over k such that G is given by a root datum relative to T . One knows that every connected and reductive group containing a split maximal torus is split (cf. [5, Chapter XXII, Proposition 2.1]). In such a case, the normalizer N of T in G and all its connected components are defined over k and the quotient N/T is the Weyl group W of the corresponding root datum.

Lemma 3.7. *Let G be a connected reductive group and let T be a split maximal torus of G contained in a Borel subgroup B . Put $H = G/N$, where N is the normalizer of T in G . Then any étale locally trivial H -fibration $f : X \rightarrow Y$ induces an isomorphism of higher Chow groups*

$$f^* : CH^*(Y, \cdot) \xrightarrow{\cong} CH^*(X, \cdot).$$

Proof. We prove the lemma in several steps. In the first step, we show that the natural map

$$(3.8) \quad CH^*(k, \cdot) \rightarrow CH^*(H, \cdot)$$

is an isomorphism.

Since N is a closed subgroup of G defined over k , it follows from [25, Theorem 12.2.1] that the quotient H is defined over k . If W denotes the Weyl group of G , then the fibration sequence

$$0 \rightarrow W \rightarrow G/T \rightarrow H \rightarrow 0$$

together with Corollary 8.3 give an isomorphism

$$(3.9) \quad CH^*(H, \cdot) \cong (CH^*(G/T, \cdot))^W.$$

Moreover, the characteristic filtration $B^u = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n = \{1\}$ of the unipotent radical B^u of B (cf. proof of Theorem 3.3) has the property that the map $G/TU_i \rightarrow G/TU_{i-1}$ is a torsor for the vector group U_{i-1}/U_i for $0 \leq i \leq n$. The homotopy invariance then implies that the map $CH^*(G/TU_{i-1}, \cdot) \rightarrow CH^*(G/TU_i, \cdot)$ is an isomorphism. Using this isomorphism for $i = 0, \dots, n$, we get an isomorphism

$$CH^*(H, \cdot) \cong (CH^*(G/T, \cdot))^W \cong (CH^*(G/B, \cdot))^W.$$

However, as G/B is a linear variety, the natural map

$$CH^*(G/B, 0) \otimes_{\mathbb{Q}} CH^*(k, \cdot) \rightarrow CH^*(G/B, \cdot)$$

is an isomorphism (cf. [18]). In particular, we get

$$CH^*(H, \cdot) \cong (CH^*(G/B, \cdot))^W \cong (CH^*(G/B, 0))^W \otimes_{\mathbb{Q}} CH^*(k, \cdot).$$

The proof of 3.8 now follows using the isomorphism $(CH^*(G/B, 0))^W \cong \mathbb{Q}$ (loc. cit.).

Now we consider the case when $X = Y \times H$ and f is the projection map. Since $G/T \times Y$ is a principal W -bundle over $H \times Y = X$, the same argument as above shows that there is an isomorphism

$$CH^*(X, \cdot) \cong CH^*(G/B \times Y, i)^W \cong CH^*(G/B, \cdot)^W \otimes_{CH^*(k, \cdot)} CH^*(Y, \cdot) \cong CH^*(Y, \cdot),$$

where the second isomorphism follows from Lemma 3.6 and the last isomorphism follows from 3.9 followed by 3.8. We finally prove the lemma by the Noetherian induction on Y . We can assume Y to be reduced. If Y is 0-dimensional, then the lemma follows from 3.8. So we assume that $\dim(Y) \geq 1$ and the lemma holds for when f is restricted to all proper closed subvarieties of Y .

Since f is étale locally trivial, there is an étale cover $\pi : Y' \rightarrow Y$ such that $X' = X \times_Y Y'$ is isomorphic to $H \times Y'$ and the pull-back of f to X' is the projection map $f' : X \times_Y Y' \rightarrow Y'$. Moreover, as π is dominant and generically finite, there exists an open set $U \subset Y$ such that $\pi_U : U' = \pi^{-1}(U) \rightarrow U$ is finite and étale. Letting $X_U = f^{-1}(U)$, we thus get a fiber square

$$\begin{array}{ccc} X'_U & \xrightarrow{f'_U} & U' \\ \pi'_U \downarrow & & \downarrow \pi_U \\ X_U & \xrightarrow{f_U} & U \end{array}$$

where $X'_U = H \times U'$ and f'_U is the projection map. Since π_U is finite étale and f_U is smooth, we get a commutative diagram (cf. Proposition 2.2)

$$\begin{array}{ccccc} CH^*(U, \cdot) & \xrightarrow{\pi_U^*} & CH^*(U', \cdot) & \xrightarrow{\pi_{U*}} & CH^*(U, \cdot) \\ \downarrow f_U^* & & \downarrow f'^*_U & & \downarrow f_U^* \\ CH^*(X_U, \cdot) & \xrightarrow{\pi'^*_U} & CH^*(X'_U, \cdot) & \xrightarrow{\pi'_{U*}} & CH^*(X_U, \cdot) \end{array}$$

and the projection formula implies that $\pi_{U*} \circ \pi_U^*$ and $\pi'_{U*} \circ \pi'^*_U$ are both multiplication by the degree of π_U . On the other hand, we have just shown above that the middle vertical arrow is an isomorphism. Hence f_U^* is also an isomorphism.

Now we let $Z = Y - U$ and put $X_Z = f^{-1}(Z)$. This gives the following commutative diagram of fibration sequences of cycle complexes.

$$\begin{array}{ccccc} \mathcal{Z}^j(Z, \cdot) & \rightarrow & \mathcal{Z}^j(Y, \cdot) & \rightarrow & \mathcal{Z}^j(U, \cdot) \\ \downarrow f_Z^* & & \downarrow f^* & & \downarrow f_U^* \\ \mathcal{Z}^j(X_Z, \cdot) & \rightarrow & \mathcal{Z}^j(X, \cdot) & \rightarrow & \mathcal{Z}^j(X_U, \cdot) \end{array}$$

We have just now shown that f_U^* is a quasi-isomorphism. The left vertical arrow is a quasi-isomorphism by the Noetherian induction. We conclude that the middle vertical arrow is also a quasi-isomorphism. \square

The following theorem is an analogue of a result of Merkurjev (cf. [22, Proposition 8]) about the equivariant K -theory. However, this result for the equivariant higher Chow groups has an advantage over Merkurjev's theorem in that it holds for the action of any split reductive group (though with rational coefficients) whereas [22, Proposition 8] is known only for the groups whose derived subgroups are simply connected, e.g., $GL_n(k)$.

Theorem 3.8. *Let G be a connected and split reductive group and let T be a split maximal torus of G . Then for any $X \in \mathcal{V}_G$, the natural map of $S(T)$ -modules*

$$CH_G^*(X, \cdot) \otimes_{S(G)} S(T) \rightarrow CH_T^*(X, \cdot)$$

is an isomorphism.

Proof. It is easy to see using the rules of the intersection product that the above map is $S(T)$ -linear, where the $S(T)$ -module structure on the left is given by the extension of scalars. So we only need to prove that this map is an isomorphism of abelian groups. Let N denote the normalizer of T in G and let $W = N/T$ be the Weyl group. If (V, U) is good pair for the action of G , then $X_N \rightarrow X_G$ is an étale locally trivial G/N -fibration. Hence it follows from Lemma 3.7 that the map

$$(3.10) \quad CH_G^*(X, \cdot) \rightarrow CH_N^*(X, \cdot)$$

is an isomorphism. In particular, we have $S(G) \cong S(N)$. Thus we only need to show that the natural map

$$(3.11) \quad CH_N^*(X, \cdot) \otimes_{S(N)} S(T) \rightarrow CH_T^*(X, \cdot)$$

is an isomorphism.

We give the Weyl group W a reduced induced scheme structure and let r be the cardinality of W . Then $CH^*(W, 0) \cong CH^*(k, 0)^{\oplus r} \cong \mathbb{Q}^r$ with a basis $\{u_1, \dots, u_r\}$. It suffices then to prove that if (V, U) is a good pair for the action of N such that it acts freely on $Y = X \times U$ and if $Y/T \xrightarrow{f} Y/N$ is the flat map, then the natural map

$$(3.12) \quad CH^*(Y/N, \cdot) \otimes_{\mathbb{Q}} CH^*(W, 0) \xrightarrow{\tilde{f}^*} CH^*(Y/T, \cdot)$$

$$(a_1, \dots, a_r) \mapsto \sum_{i=1}^r f^*(a_i)$$

is an isomorphism. For, this would imply that $S(N) \otimes_{\mathbb{Q}} CH^*(W, 0) \xrightarrow{\cong} S(T)$ and $CH_N^*(X, \cdot) \otimes_{\mathbb{Q}} CH^*(W, 0) \cong CH_T^*(X, \cdot)$. But the left term of this isomorphism is same as

$$(CH_N^*(X, \cdot) \otimes_{S(N)} S(N)) \otimes_{\mathbb{Q}} CH^*(W, 0) \cong CH_N^*(X, \cdot) \otimes_{S(N)} S(T).$$

In order to prove 3.12, let $Z = Y/N$ and $Z' = Y/T$. Since $Z' \xrightarrow{f} Z$ is a principal W -bundle, one has a fiber diagram

$$\begin{array}{ccc} Z' \times W & \xrightarrow{\pi} & Z' \\ \pi \downarrow & & \downarrow f \\ Z' & \xrightarrow{f} & Z \end{array}$$

This gives a commutative diagram

$$\begin{array}{ccccc} CH^*(Z, \cdot) \otimes CH^*(W, 0) & \xrightarrow{f^* \otimes id} & CH^*(Z', \cdot) \otimes CH^*(W, 0) & \xrightarrow{f_* \otimes id} & CH^*(Z, \cdot) \otimes CH^*(W, 0) \\ \downarrow \tilde{f}^* & & \downarrow \pi^* & & \downarrow \tilde{f}^* \\ CH^*(Z', \cdot) & \xrightarrow{\pi^*} & CH^*(Z' \times W, \cdot) & \xrightarrow{\pi_*} & CH^*(Z', \cdot) \end{array}$$

where the tensor product in the top row is over \mathbb{Q} . The right (and the left) vertical map is given by $\tilde{f}^*(a_1, \dots, a_r) = \sum_{i=1}^r f^*(a_i)$ and so is the middle vertical map. Moreover, the composite horizontal arrows are identity. Hence \tilde{f}^* is an isomorphism as it is a retract of the middle vertical arrow, which is clearly an isomorphism. This proves 3.12 and hence the theorem. \square

Corollary 3.9. *Let G be a connected and split reductive group and let T be a split maximal torus of G with the Weyl group W . Then for any $X \in \mathcal{V}_G$, the restriction map r_T^G induces an isomorphism*

$$CH_G^*(X, \cdot) \xrightarrow{\cong} (CH_T^*(X, \cdot))^W.$$

Proof. This follows directly by taking the Weyl group invariants on the both sides of the isomorphism in Theorem 3.8 and then using the isomorphism $S(G) = S(T)^W$ (cf. [7, Proposition 6]). \square

We end this section with the following structure theorem for the equivariant higher Chow groups of a variety with the action of a diagonalizable group on which certain subgroup acts trivially. An analogous result for the equivariant K -theory for the torus action was proved by Thomason (cf. [29, Lemma 5.6]).

Theorem 3.10. *Let T be a split diagonalizable group and let $X \in \mathcal{V}_T$. Let H be a connected closed subgroup of T which acts trivially on X . Then there is a natural isomorphism*

$$CH_{T/H}^*(X, \cdot) \otimes_{\mathbb{Q}} S(H) \xrightarrow{i_H^T} CH_T^*(X, \cdot).$$

Proof. Put $T' = T/H$. Since H is a torus, we can choose a decomposition (not necessarily canonical) $T = H \times T'$. Fix an integer $j \geq 0$ and let (V, U) and (V', U') be good pairs for the actions of H and T' respectively corresponding to j as in [7, Example 3.1]. Thus U is a product of punctured affine spaces and $U/H = (\mathbb{P}^n)^r$ for some $n \gg 0$, where $r = \text{rank}(H)$. Then (V_T, U_T) with $V_T = V \times V'$ and $U_T = U \times U'$, is a good pair for j for the action of T . Note that $CH_T^j(X, \cdot)$ does not depend on the choice of the decomposition of T since it does not depend on the choice of the good pair (V_T, U_T) . Now we have

$$X_T = (X \times U \times U') / (H \times T') = (X \times U') \times^{T'} U/H = X_{T'} \times (\mathbb{P}^n)^r,$$

where the second equality holds since H acts trivially on $X \times U'$ and the third equality holds because T' acts trivially on U . We now apply the projective bundle formula for the non-equivariant higher Chow groups to conclude that the map

$$\bigoplus_{p+q=j} CH^p(X_{T'}, \cdot) \otimes_{\mathbb{Q}} CH^q((\mathbb{P}^n)^r, 0) \rightarrow CH^j(X_T, \cdot)$$

is an isomorphism. However, $CH^p(X_{T'}, \cdot) \cong CH_{T'}^p(X, \cdot)$ for all $p \leq j$ and $CH^j(X_T, \cdot) \cong CH_T^j(X, \cdot)$. This finishes the proof. \square

4. COMPLETIONS OF EQUIVARIANT HIGHER CHOW GROUPS

Let G be a linear algebraic group. Recall that the *Chow ring* $S(G)$ of G is the graded ring $CH_G^*(k, 0) = \bigoplus_{j=0}^{\infty} CH_G^j(k, 0)$, and $\widehat{S(G)}$ is the J_G -adic completion of $S(G)$, where J_G is the augmentation ideal of cycles of positive codimension. For $X \in \mathcal{V}_G$, we can define various completions of the graded $S(G)$ -module $CH_G^*(X, \cdot)$. Our objective in this section is to analyze the relation between these completions. Our main ingredient for this analysis is the following general algebraic result.

Let R_0 be a commutative Noetherian ring and let $R = \bigoplus_{j=0}^{\infty} R_j$ be a finitely generated graded R_0 -algebra. Let $I = \bigoplus_{j=1}^{\infty} R_j$ be the irrelevant ideal of R . Let $M = \bigoplus_{j=0}^{\infty} M_j$ be a graded R -module which need not be finitely generated. Put $M^i = \bigoplus_{j=i}^{\infty} M_j$ for $i \geq 0$. Then M^\cdot defines a filtration of M by R -submodules. We shall call this the *filtration of M by grading*. Let \widehat{R} be the I -adic completion of R . Associated to M , we define the following three modules.

$\widetilde{M} = M \otimes_R \widehat{R}$ (the weak completion of M)
 $\widehat{M} = \widehat{M_I}$ (the I -adic completion of M)
 \overline{M} = the completion defined by the filtration M^\cdot
 (the graded completion of M).

Note that \overline{M} is an \widehat{R} -submodule of the product $\prod_{i=0}^{\infty} M/M^i$ and the natural map

$$(4.1) \quad \prod_{i=0}^{\infty} M_i \longrightarrow \prod_{i=0}^{\infty} M/M^i$$

$$(m_i) \mapsto ((m_0, \dots, m_{i-1}))$$

identifies $\prod_{i=0}^{\infty} M_i$ with \overline{M} . Moreover, the map $M \rightarrow \overline{M}$ is the natural embedding of the direct sum into the direct product. All the above completions of M are \widehat{R} -modules and there are natural maps

$$(4.2) \quad M \longrightarrow \widetilde{M} \xrightarrow{\phi_M} \widehat{M} \xrightarrow{\theta_M} \overline{M},$$

$$\psi_M$$

where ϕ_M and θ_M (hence their composite ψ_M) are \widehat{R} -linear.

Proposition 4.1. *For the graded ring R , the following hold.*

- (i) *The graded completion is an exact functor on the category of graded R -modules.*
- (ii) *ϕ_M is an isomorphism if M is finitely generated graded R -module.*
- (iii) *θ_M is an isomorphism if M is generated as R -module by a (possibly infinite) set S of homogeneous elements of bounded degree.*
- (iv) *ψ_M (hence ϕ_M) is injective for any graded R -module M .*
- (v) *ψ_M need not be surjective for any graded R -module M .*

Proof. To prove (i), we note that a sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

of graded R -linear maps is exact if and only if

$$0 \rightarrow M'_j \rightarrow M_j \rightarrow M''_j \rightarrow 0$$

is exact sequence of R_0 -modules for every $j \geq 0$. Equivalently, the sequence

$$0 \rightarrow \prod_{j=0}^{\infty} M'_j \rightarrow \prod_{j=0}^{\infty} M_j \rightarrow \prod_{j=0}^{\infty} M''_j \rightarrow 0$$

is exact, which proves (i). The part (ii) is obvious since R is Noetherian.

For (iii), it suffices to show that the I -adic filtration and the filtration by grading give the same topology on M . We already have $I^i M \subseteq M^i$ for every $i \geq 0$. So we need to prove that for given $n \geq 1$, one has $M^i \subseteq I^n M$ for all $i \gg 0$. Since R is a finitely generated R_0 -algebra, we can assume that there exists a finite set T of homogeneous elements of positive degree in R which generate R as an R_0 -algebra. Let N denote the maximum of the bounded degrees of the sets S and T , and the cardinality of T . It suffices then to show that

$$(4.3) \quad M^i \subseteq I^n M \text{ for } i \geq N' = (N^2 n + 1)N.$$

So let $m \in M_j$ with $j \geq i \geq N'$ and write $m = \sum_{u=0}^r a_u m_u$ with $\deg(m_u) \leq N$. Then for every u , we must have $j = \deg(a_u) + \deg(m_u) \geq N'$, which implies that $\deg(a_u) \geq N' - \deg(m_u) \geq N' - N = N^2 n$. This shows that

$$M^i \subseteq R_{\geq N^2 n} M.$$

Thus it suffices to show that $R_{\geq N^2 n} \subseteq I^n$ in order to prove 4.3. So let $a \in R_j$ with $j \geq N^2 n$ and write $a = a_1^{t_1} \cdots a_r^{t_r}$ with $a_u \in T$. We need to show that $\sum_{u=1}^r t_u \geq n$ (which would imply that $a \in I^n$). However, otherwise we would get

$$N^2 n \leq j = \sum_{u=1}^r \deg(a_u) t_u < \sum_{u=1}^r N n \leq N^2 n,$$

which is absurd. This proves (iii).

To prove (iv), let M be a graded R -module. Then there exists a direct system $\{M_\lambda\}$ of finitely generated graded R -submodules of M such that $\varinjlim_\lambda M_\lambda \xrightarrow{\cong} M$, which in turn gives $\varinjlim_\lambda (M_\lambda \otimes_R \widehat{R}) \xrightarrow{\cong} M \otimes_R \widehat{R}$. This gives us a commutative diagram

$$\begin{array}{ccc} \varinjlim_\lambda (M_\lambda \otimes_R \widehat{R}) & \longrightarrow & \varinjlim_\lambda \overline{M_\lambda} \\ \downarrow & & \downarrow \\ M \otimes_R \widehat{R} & \xrightarrow{\psi_M} & \overline{M}. \end{array}$$

The top horizontal arrow is an isomorphism by the parts (ii) and (iii) of the proposition. We have just seen that the left vertical arrow is an isomorphism. The right vertical arrow is injective by using the exactness of the graded completion and the direct limit functors, plus the fact that each $M_\lambda \hookrightarrow M$. Hence the bottom horizontal arrow must be injective.

To see (v), take $M = R^{\mathbb{N}}$. Then it is easy to check that $\widetilde{M} = \left(\prod_{j=0}^{\infty} R_j \right)^{\mathbb{N}}$ whereas $\overline{M} = \prod_{j=0}^{\infty} R_j^{\mathbb{N}}$. Hence ψ_M is not surjective. \square

We need a few intermediate results before we give our first application of Proposition 4.1 to the equivariant higher Chow groups.

Lemma 4.2. *Let G be a linear algebraic group over k such that the unipotent radical $R_u G$ is defined over k . Let $L = G/R_u G$ be the reductive quotient of G . Then there are natural isomorphisms $R(L) \xrightarrow{\cong} R(G)$ and $S(L) \xrightarrow{\cong} S(G)$.*

Proof. Since $R_u G$ is defined over k , it follows from [25, Corollary 12.2.2] that the reductive quotient L is also defined over k . Let V be an irreducible representation of G . Then the fixed point theorem (cf. [17, Theorem 17.5]) implies that if

the unipotent radical $R_u G$ acts non-trivially on V , then the invariance subspace $W \subset V$ is non-zero. Moreover, W is then G -invariant which contradicts the irreducibility of V . We conclude that $R_u G$ acts trivially on V . Since $R(G)$ is a free abelian group on the set of irreducible representations of G , we see that the map $R(L) \rightarrow R(G)$ must be an isomorphism.

For proving the corresponding isomorphism of the Chow rings, we fix any $j \geq 0$ and choose good pairs (V_1, U_1) and (V_2, U_2) for G and L respectively, and let G act on $V = V_1 \oplus V_2$ diagonally where it acts on V_2 via L . Then $(V_1 \times V_2, U_1 \times U_2)$ is a good pair for G . Moreover, the G -equivariant projection map $U_1 \times U_2 \rightarrow U_2$ induces the map on quotients $(U_1 \times U_2)/G \rightarrow U_2/G = U_2/L$. This gives the pull-back map on Chow groups $CH_L^j(k, i) = CH^j(U_2/L, i) \rightarrow CH^j((U_1 \times U_2)/G, i) = CH_G^j(k, i)$. This gives the natural map $S(L) \rightarrow S(G)$. To show that this is an isomorphism, we use the isomorphism $R(L) \xrightarrow{\cong} R(G)$ shown above and the following commutative diagram of completions (cf. [6, Lemma 3.2]).

$$\begin{array}{ccccc} \widehat{R(L)} & \longrightarrow & \prod_{j \geq 0} CH_L^j(k) & \longleftarrow & S(L) \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{R(G)} & \longrightarrow & \prod_{j \geq 0} CH_G^j(k) & \longleftarrow & S(G) \end{array}$$

The horizontal arrows in the left square are ring isomorphisms by Theorem 1.1 and we have shown above that the left vertical arrow is an isomorphism of rings. Thus the middle vertical arrow is also an isomorphism. The horizontal arrows in the right square are clearly injective. We conclude from this that the right vertical arrow is injective. On the other hand, the isomorphism of the middle vertical arrow and the natural surjection $\prod_{j \geq 0} CH_G^j(k) \twoheadrightarrow CH_G^j(k)$ implies that the map

$CH_L^j(k) \rightarrow CH_G^j(k)$ is surjective for each $j \geq 0$ and hence, the map $S(L) \rightarrow S(G)$ is also surjective and consequently an isomorphism. \square

Remark 4.3. We remark here that isomorphism of $S(L) \rightarrow S(G)$ in characteristic zero follows directly from Proposition 3.4. The above indirect argument is given to take care of the positive characteristic case when G might not have Levi subgroups.

Lemma 4.4. *Let G be a linear algebraic group acting on a quasi-projective variety X over k . Let l/k be a finite extension of k and let $CH_G^*(X_l, i)$ denote the equivariant higher Chow groups for the action of the group G_l on X_l . Then there are natural maps $CH_G^*(X, i) \rightarrow CH_G^*(X_l, i) \rightarrow CH_G^*(X, i)$ such that the composite map is multiplication by the degree $[l : k]$ of the field extension.*

Proof. Fix $j \geq 0$ and let (V, U) be a good pair for the G -action corresponding to j . Since the map $U \rightarrow U/G$ is a principal G -bundle of quasi-projective k -schemes, we see that $U_l \rightarrow (U/G)_l$ is a principal G_l -bundle. In particular, (V_l, U_l) is a good pair for the G_l -action. Similarly, the principal G -bundle $X \times U \rightarrow X_G$ implies that $X_l \times_l U_l \rightarrow (X_G)_l$ is a principal G_l -bundle, which shows that the mixed quotient

$X_l \times^{G_l} U_l$ is isomorphic to $(X_G)_l$. Using this, we get

$$(4.4) \quad CH_G^j(X_l, i) \cong CH^j\left(X_l \times^{G_l} U_l, i\right) \cong CH^j((X_G)_l, i).$$

Let $\pi : (X_G)_l \rightarrow X_G$ denote the natural finite map. Then the flatness of π implies that there are pull-back and push-forward maps $CH^j(X_G, i) \xrightarrow{\pi^*} CH^j((X_G)_l, i) \xrightarrow{\pi_*} CH^j(X_G, i)$ such that $\pi_* \circ \pi^*$ is multiplication by $[l : k]$ by [2, Corollary 1.4]. The proof is completed by combining this with the isomorphism in 4.4. \square

Remark 4.5. It is easy to see from the above proof that if l/k is a Galois extension in Lemma 4.4, then $CH_G^*(X, i)$ is in fact the Galois invariant of $CH_G^*(X_l, i)$ under π^* and π_* is just the trace map. This follows for example, from [14, Lemma 8.4] and the non-equivariant Riemann-Roch isomorphism of [2].

We next present the following two elementary results in commutative algebra which are useful in describing when a subring of a Noetherian ring is also Noetherian. Recall that if B is a commutative ring containing a subring A , then one says that A is a *pure subring* of B , if for any A -module M , the natural map $M \rightarrow M \otimes_A B$ is injective. One example of pure subrings which often occurs is the case when A is a retract of B as an A -module. It is known that pure subrings share many good properties of the ambient ring. We mention here one such property that will be useful in this paper.

Lemma 4.6. *Let R be a Noetherian ring, B a finitely generated R -algebra, and A a pure R -subalgebra of B . Then A is finitely generated over R .*

Proof. Cf. [16, Theorem 1]. \square

Lemma 4.7. *Let $R \subset S$ be an inclusion of commutative rings such that R is Noetherian and S is finitely generated as R -algebra. Let G be a finite group of R -algebra automorphisms of S . Then the ring of invariants S^G is also a finitely generated R -algebra, and hence Noetherian.*

Proof. By [1, Proposition 7.8], one only needs to show that S is integral over S^G . So let $s \in S$ and put

$$f(s) = \prod_{\sigma \in G} (s - \sigma(s)).$$

Then $f(s)$ is clearly zero (take $\sigma = 1$ on the right). On the other hand, it is easy to see that $f(s)$ is a monic polynomial in s of degree equal to the cardinality of G and with coefficients in S^G . \square

Let G be a linear algebraic group over k and let $X \in \mathcal{V}_G$. In the rest of this paper, we shall follow the notations for the various completions defined in the beginning of this section for the ring $S(G)$ and the graded $S(G)$ -module $CH_G^*(X, \cdot)$.

Corollary 4.8. *Let G and $X \in \mathcal{V}_G$ be as above. Then the natural maps of $\widehat{S(G)}$ -modules*

$$\begin{aligned} \widehat{CH_G^*(X, \cdot)} &\rightarrow \widehat{CH_G^*(X, \cdot)} \quad \text{and} \\ \widehat{CH_G^*(X, \cdot)} &\rightarrow \overline{CH_G^*(X, \cdot)} \end{aligned}$$

are injective.

Proof. By Proposition 4.1, we only need to show that $S(G)$ is a finitely generated \mathbb{Q} ($= S(G)_0$)-algebra. Now, there is a finite extension l/k such that all the connected components of algebraic group G_l are defined over l , the identity component G_l^0 is split and its unipotent radical $R_u(G_l^0)$ is also defined and split over l . By applying Lemma 4.4 to $X = \text{Spec}(k)$, we see that there are pull-back and push-forward maps $S(G) \rightarrow S(G_l) \rightarrow S(G)$ such that the composite map is multiplication by the degree $[l : k]$. Moreover, the commutativity of the intersection product with the pull-back, and the projection formula (cf. Proposition 2.2) show that these maps are $S(G)$ -linear. We conclude that $S(G)$ is a retract of $S(G_l)$ as an $S(G)$ -module, and hence a pure \mathbb{Q} -subalgebra. By Lemma 4.6, it suffices to show that $S(G_l)$ is a finitely generated \mathbb{Q} -algebra. Hence we can assume that G has all the properties described in the beginning of the proof.

Using Corollary 8.3 and Lemma 4.7, we can assume that G is connected. By Lemma 4.2, we can further assume that G is reductive. Since G is split, we can now use Corollary 3.9 and Lemma 4.7 once again to reduce to the case when G is a split torus. But then $S(G)$ is known to be a finitely generated polynomial algebra over \mathbb{Q} . \square

5. COHOMOLOGICAL RIGIDITY AND SPECIALIZATIONS

Let G be a split diagonalizable group over k acting on a smooth variety X . Recall (cf. [25, 13.2.5]) that all the diagonalizable subgroups of G are defined and split over k . The equivariant K -theory of X for the G -action was studied by Vezzosi and Vistoli in [34]. Their main result (Theorem 1) is to show how to reconstruct the K -theory ring of X in terms of the equivariant K -theory of the loci where the stabilizers have constant dimension. In the next two sections, we use the ideas of Vezzosi-Vistoli to prove an analogous decomposition theorem for the equivariant higher Chow groups of X for the G -action. This theorem and its compatibility with the corresponding result for K -theory will be crucial in the proof of the main results of this paper. Like in the case of K -theory (*loc. cit.*), this decomposition can also be used to compute the equivariant higher Chow groups of many varieties with an action of the group G such as the toric varieties. This section is concerned with the study of cohomological rigidity and the construction of the specialization maps in equivariant higher Chow groups. For the rest of this section and the next, the group G will always denote a split diagonalizable group and the varieties will be assumed to be smooth with G -action. We have seen (cf. Proposition 2.2) that for such a variety X , $CH_G^*(X, \cdot)$ is a bigraded ring which is an algebra over the ring $CH_G^*(k, \cdot)$. We denote the full equivariant Chow ring $\bigoplus_{j,i \geq 0} CH_G^j(X, i)$ of X in short by $CH_G^*(X)$.

5.1. Cohomological rigidity.

Definition 5.1. Let $Y \subset X$ be a smooth and G -invariant closed subvariety of codimension $d \geq 1$ and let $N_{Y/X}$ denote the normal bundle of Y in X . We say that Y is *cohomologically rigid* inside X if $c_d^G(N_{Y/X})$ is not a zero-divisor in the ring $CH_G^*(Y)$.

As one observes, this definition has any reasonable meaning only in the equivariant setting, since every element of positive degree in the non-equivariant Chow ring is nilpotent. The importance of cohomological rigidity for the equivariant higher Chow groups comes from the following analogue of the K -theory splitting theorem (Proposition 4.3) of *loc. cit.*.

Proposition 5.2. *Let Y be a smooth and G -invariant closed subvariety of X of codimension $d \geq 1$. Assume that Y is cohomologically rigid inside X , and put*

$U = X - Y$. Let $Y \xrightarrow{i} X$ and $U \xrightarrow{j} X$ be the inclusion maps. Then

(i) The localization sequence

$$0 \rightarrow CH_G^*(Y) \xrightarrow{i_*} CH_G^*(X) \xrightarrow{j^*} CH_G^*(U) \rightarrow 0$$

is exact.

(ii) The restriction ring homomorphisms

$$CH_G^*(X) \xrightarrow{(i^*, j^*)} CH_G^*(Y) \times CH_G^*(U)$$

give an isomorphism of rings

$$CH_G^*(X) \xrightarrow{\cong} CH_G^*(Y) \times_{\widetilde{CH_G^*(Y)}} CH_G^*(U),$$

where $\widetilde{CH_G^(Y)} = CH_G^*(Y)/(c_d^G(N_{Y/X}))$, and the maps*

$$CH_G^*(Y) \rightarrow \widetilde{CH_G^*(Y)}, \quad CH_G^*(U) \rightarrow \widetilde{CH_G^*(Y)}$$

are respectively, the natural surjection and the map

$$CH_G^*(U) = \frac{CH_G^*(X)}{i_*(CH_G^*(Y))} \xrightarrow{i^*} \frac{CH_G^*(Y)}{c_d^G(N_{Y/X})} = \widetilde{CH_G^*(Y)},$$

which is well-defined by Corollary 2.6.

Proof. The part (i) follows directly from Corollary 2.6 and the definition of cohomological rigidity. Since i^* and j^* are ring homomorphisms, the proof of the second part follows directly from the first part and [loc. cit., Lemma 4.4]. \square

To apply the above result in our context, we need to have some sufficient conditions for checking the cohomological rigidity in specific examples. We first have the following elementary result.

Lemma 5.3. *Let A be a ring which is a \mathbb{Q} -algebra. Then an element of the form t^d , where $t = \sum_{j=1}^r a_j t_{i_j} \in A[t_1, \dots, t_n]$, is not a zero-divisor for every $d \geq 1$, whenever $a_j \in \mathbb{Q}$ for all j and $a_j \neq 0$ for some j .*

Proof. We can assume that $a_j \neq 0 \forall j$. We prove by induction on n . For $n = 1$, it is obvious. So we assume that the lemma holds for all $n' \leq n - 1$ and $n \geq 2$.

If $r = 1$, then also the lemma is again obvious. So we assume $r \geq 2$. Let $\underline{t} = (t_1, \dots, t_n)$, and let $f(\underline{t}) = f_0(\underline{t}) + \dots + f_p(\underline{t})$ be a polynomial such that each f_i is homogeneous of degree d_i such that $0 \leq d_0 < \dots < d_p$. If $f(\underline{t}) \neq 0$ and $t^d f(\underline{t}) = 0$, then using the fact that t^d is homogeneous, it is easy to see that

$t^d f_q(\underline{t}) = 0$, where d_q is the largest integer such that $f_q \neq 0$. Thus we can assume that $f(\underline{t}) \neq 0$ is homogeneous of degree, say p .

Let $f(\underline{t}) = b_1 f_1 + \cdots + b_m f_m$ be the unique representation of $f(\underline{t})$ as an A -linear combination of linearly independent monomials f_i 's of homogeneous degree p . Let $s \geq 0$ be the largest integer such that t_1^s divides each f_i . Then $t^d f(\underline{t}) = t^d t_1^s (b_1 f'_1 + \cdots + b_m f'_m)$, where some f'_j , say f'_1 is not divisible by t_1 . Now by $r = 1$ case, $t^d f(\underline{t}) = 0$ implies that $t^d f'(\underline{t}) = 0$, where $f'(\underline{t}) = \sum_{j=1}^m b_j f'_j$. For any element $g \in A[t_1, \dots, t_n]$, let \bar{g} denote its image in the quotient $A[t_2, \dots, t_n]$. Then we get $\bar{t}^d \bar{f}' = 0$ in $A[t_2, \dots, t_n]$. However, $r \geq 2$ implies that $\bar{t} \neq 0$ and $\bar{f}'_1 \neq 0$ by our choice. By induction on n , this leads to a contradiction. \square

Proposition 5.4. *Let G be a split diagonalizable group acting on a smooth variety X and let E be a G -equivariant vector bundle of rank d on X . Assume that there is a subtorus $T \subset G$ of positive rank which acts trivially on X , such that in the eigenspace decomposition of E with respect to T , the submodule corresponding to the trivial character is zero. Then $c_d^G(E)$ is not a zero-divisor in $CH_G^*(X)$.*

Proof. By [29, Lemma 5.6], E has a unique direct sum decomposition

$$E = \bigoplus_{i=1}^r E_{\chi_i} \otimes \chi_i,$$

where we choose a splitting $G = D \times T$, and E_{χ_i} 's are D -bundles and χ_i 's are 1-dimensional representations of T . This decomposition is via the functor

$$\begin{aligned} \text{Bun}_X^D \times \text{Rep}(T) &\rightarrow \text{Bun}_X^G \\ (F, \rho) &\mapsto p_1^*(F) \otimes p_2^*(\rho), \end{aligned}$$

where $p_1 : D \times T \rightarrow D$ and $p_2 : D \times T \rightarrow T$ are the projections.

Since $\text{rank}(E) = d$, we have by the Whitney sum formula, $c_d^G(E) = \prod_{i=1}^r c_{d_i}^G(E_{\chi_i} \otimes \chi_i)$,

where $d_i = \text{rank}(E_i)$. Thus we can assume that $E = E_\chi \otimes \chi$, where χ is not a trivial character by our assumption. In particular, if L_χ is the corresponding line bundle in $\text{Pic}_T(k)$, then

$$(5.1) \quad c_1^T(L_\chi) = t = \sum_{i=1}^p n_i t_i \in \mathbb{Q}[t_1, \dots, t_n]$$

with $n_i \neq 0$ for some i . By neglecting those i for which the coefficients n_i 's are zero, we can assume that $n_i \neq 0 \forall i$. Now we have

$$\begin{aligned} c_d^G(E) &= c_d^G(p_1^*(E_\chi) \otimes p_2^*(L_\chi)) \\ &= \sum_{i=0}^d c_{d-i}^G(p_1^*(E_\chi)) \cdot (c_1^G(p_2^*(L_\chi)))^i \\ &= \sum_{i=0}^d p_1^*(c_{d-i}^D(E_\chi)) \cdot p_2^*\left((c_1^T(L_\chi))^i\right) \\ &= \sum_{i=0}^d \alpha_i t^i, \end{aligned}$$

where $\alpha_i \in CH_D^*(X)$ and $c_d^G(E) \in CH_G^*(X) \cong CH_D^*(X) \otimes S(T)$ (cf. Theorem 3.10) and the second equality holds by [11, Remark 3.2.3]. Furthermore, one has $\alpha_d = p_1^*(c_0^D(E_X)) = 1$. Thus we get $c_d^G(E) = t^d + \alpha_{d-1}t^{d-1} + \cdots + \alpha_1t + \alpha_0 = g(t)$.

We need to show that $g(t)$ is not a zero divisor in $CH_D^*(X)[t_1, \dots, t_n]$. So suppose $f(\underline{t})$ is a non-zero polynomial such that $g(t)f(\underline{t}) = 0$, and let $f'(\underline{t})$ be the homogeneous part of $f(\underline{t})$ of largest degree which is not zero. By comparing the homogeneous parts, it is easy to see that $g(t)f(\underline{t}) = 0$ only if $t^d f'(\underline{t}) = 0$. But this is a contradiction since t satisfies the condition of Lemma 5.3 by 5.1, and hence is not a zero-divisor. \square

Let G be a split diagonalizable group as above and let $X \in \mathcal{V}_G^S$. Following the notations of [34], for any $s \geq 0$, we let $X_{\leq s} \subset X$ be the open subset of points whose stabilizers have dimension at most s . We shall often write $X_{\leq s-1}$ also as $X_{< s}$. Let $X_s = X_{\leq s} - X_{< s}$ denote the locally closed subset of X , where the stabilizers have dimension exactly s . We think of X_s as a subspace of X with the reduced induced structure. It is clear that $X_{\leq s}$ and X_s are G -invariant subspaces of X . Let N_s denote the normal bundle of X_s in $X_{\leq s}$, and let N_s^0 denote the complement of the 0-section in N_s . Then G clearly acts on N_s . The following result of *loc. cit.* describes some properties of these subspaces which will be useful to us in what follows.

Proposition 5.5 ([34]). *Let s be non-zero integer.*

- (i) *There exists a finite number of s -dimensional subtori T_1, \dots, T_r in G such that X_s is the disjoint union of the fixed point spaces $X_{\leq s}^{T_j}$.*
- (ii) *X_s is smooth locally closed subvariety of X .*
- (iii) *$N_s^0 = (N_s)_{< s}$.*

Proof. See [loc. cit., Proposition 2.2]. \square

Remark 5.6. We mention here that although the above proposition has been stated for the smooth varieties, the part (i) of the proposition holds also when X is not necessarily smooth, since the proof only uses Thomason's generic étale slice theorem [27, Proposition 4.10] which holds very generally.

We have the following important application of Proposition 5.4.

Corollary 5.7. *For $s \geq 1$, X_s is cohomologically rigid inside $X_{\leq s}$.*

Proof. Let d_s be the codimension of X_s in $X_{\leq s}$. We need to show that $c_{d_s}^G(N_s)$ is not a zero-divisor in $CH_G^*(X_s)$. By Proposition 5.4, it suffices to show that there exists a subtorus T in G of positive rank which acts trivially on X_s , such that in the eigenspace decomposition of N_s with respect to T , the submodule corresponding to the trivial character is zero. But this follows directly from the parts (i) and (iii) of Proposition 5.5 and the fact that $s \geq 1$ (see [loc. cit., Proposition 4.6]). \square

5.2. Specialization maps. Let G and X be as above, and let n be the dimension of G . As seen above, there is a filtration of X by G -invariant open subsets $\emptyset = X_{\leq -1} \subset X_{\leq 0} \subset \cdots \subset X_{\leq n} = X$. In particular, G acts on $X_{\leq 0}$ with finite stabilizers, and the toral component of G acts trivially on X_n . We fix $1 \leq s \leq n$ and let $X_s \xrightarrow{f_s} X_{\leq s}$ and $X_{< s} \xrightarrow{g_s} X_{\leq s}$ denote the closed and the open embeddings

respectively. Let $\pi : M_s \rightarrow \mathbb{P}^1$ be the deformation to the normal cone for the embedding f_s as in Section 2. We have already observed there that for the trivial action of G on \mathbb{P}^1 , M_s has a natural G -action. Moreover, the deformation diagram 2.3 is a diagram of smooth G -spaces. For $0 \leq t \leq s$, we shall often denote the open subspace $(M_s)_{\leq t}$ of M_s by $M_{s,\leq t}$. The terms like $M_{s,t}$ and $M_{s,<t}$ (and also for N_s) will have similar meaning in what follows. Since G acts trivially on \mathbb{P}^1 , it acts on M_s fiberwise, and one has $N_s = \pi^{-1}(\infty)$ and

$$(5.2) \quad M_{s,\leq t} \cap N_s = N_{s,\leq t}; \quad M_{s,t} \cap N_s = N_{s,t}.$$

$$(5.3) \quad M_{s,\leq t} \cap \pi^{-1}(\mathbb{A}^1) = X_{\leq t} \times \mathbb{A}^1; \quad M_{s,t} \cap \pi^{-1}(\mathbb{A}^1) = X_t \times \mathbb{A}^1$$

$$(5.4) \quad M_{s,\leq t} \cap \pi^{-1}(\infty) = N_{s,\leq t}; \quad M_{s,t} \cap \pi^{-1}(\infty) = N_{s,t}.$$

Let $N_{s,\leq t} \xrightarrow{i_{s,\leq t}} M_{s,\leq t}$ and $X_{\leq t} \times \mathbb{A}^1 \xrightarrow{j_{s,\leq t}} M_{s,\leq t}$ denote the obvious closed and open embeddings. We define $i_{s,t}$ and $j_{s,t}$ similarly. Let $N_{s,t} \xrightarrow{\eta_{s,t}} N_{s,\leq t}$ and $M_{s,t} \xrightarrow{\delta_{s,t}} M_{s,\leq t}$ denote the other closed embeddings. One has a commutative diagram

$$(5.5) \quad \begin{array}{ccccccc} X_{\leq t} & \xrightarrow{g_{\leq t}} & X_{\leq t} \times \mathbb{A}^1 & \xrightarrow{j_{s,\leq t}} & M_{s,\leq t} & & \\ f_{s,\leq t} \downarrow & & \downarrow & & \downarrow & & \\ X_{\leq s} & \xrightarrow{g_{0,\leq s}} & X_{\leq s} \times \mathbb{A}^1 & \xrightarrow{j_{\leq s}} & M_s & \longrightarrow & X_{\leq s} \times \mathbb{P}^1 \longrightarrow X_{\leq s}, \end{array}$$

where $g_{\leq t}$ is the 0-section embedding, and the composite of all the maps in the bottom row is identity. This gives us the following diagram of equivariant higher Chow groups, where all squares commute (cf. Proposition 2.2).

$$(5.6) \quad \begin{array}{ccccccc} CH_G^*(N_{s,t}) & \xrightarrow{i_{s,t,*}} & CH_G^*(M_{s,t}) & \xrightarrow{j_{s,t}^*} & CH_G^*(X_t \times \mathbb{A}^1) & \xrightarrow{g_t^*} & CH_G^*(X_t) \\ \downarrow \eta_{s,t,*} & & \downarrow \delta_{s,t,*} & & \downarrow f_{t,*} & & \downarrow f_{t,*} \\ CH_G^*(N_{s,\leq t}) & \xrightarrow{i_{s,\leq t,*}} & CH_G^*(M_{s,\leq t}) & \xrightarrow{j_{s,\leq t}^*} & CH_G^*(X_{s,\leq t} \times \mathbb{A}^1) & \xrightarrow{g_{\leq t}^*} & CH_G^*(X_{\leq t}) \end{array}$$

Since the last horizontal maps in both rows are natural isomorphisms by the homotopy invariance, we shall often identify the last two terms in both rows and use $j_{s,\leq t}^*$ and $(j_{s,\leq t} \circ g_{\leq t})^*$ interchangeably.

Theorem 5.8. *The maps $j_{s,\leq t}^*$ and $j_{s,t}^*$ are surjective and there are ring homomorphisms*

$$\overline{Sp}_{X,s}^{\leq t} : CH_G^*(X_{\leq t}) \rightarrow CH_G^*(N_{s,\leq t});$$

$$\overline{Sp}_{X,s}^t : CH_G^*(X_t) \rightarrow CH_G^*(N_{s,t})$$

such that $i_{s,\leq t}^* = \overline{Sp}_{X,s}^{\leq t} \circ j_{s,\leq t}^*$ and $i_{s,t}^* = \overline{Sp}_{X,s}^t \circ j_{s,t}^*$. Moreover, both the squares in the diagram

$$(5.7) \quad \begin{array}{ccccc} CH_G^*(X_{\leq t}) & \xrightarrow{f_t^*} & CH_G^*(X_t) & \xrightarrow{f_{t*}} & CH_G^*(X_{\leq t}) \\ \overline{Sp}_{X,s}^{\leq t} \downarrow & & \downarrow \overline{Sp}_{X,s}^t & & \downarrow \overline{Sp}_{X,s}^{\leq t} \\ CH_G^*(N_{s,\leq t}) & \xrightarrow{\eta_{s,t}^*} & CH_G^*(N_{s,t}) & \xrightarrow{\eta_{s,t*}} & CH_G^*(N_{s,\leq t}) \end{array}$$

commute.

Proof. Using the results of this section and the previous ones, one can prove this theorem along the lines of the proof of the analogous result [*loc. cit.*, Theorem 3.2] for K -theory as given in [35]. However, it is not at all clear from the construction of the specialization maps in [35] that these maps have good functorial properties, and if they are ring homomorphisms. In particular, it is not clear if these maps will have the compatibility properties with the Chern character and Riemann-Roch maps from the equivariant K -groups to higher Chow groups. We give here a more direct and functorial construction of the specialization maps, which works both for the K -theory as well as the higher Chow groups, and the proof of various compatibilities of these maps then becomes essentially obvious. We give here the construction of these maps for the higher Chow groups. The same construction works also for the K -theory without any change.

First of all, using Corollary 5.7 and Proposition 5.2, we see that for $1 \leq s \leq n$ and $0 \leq t \leq s$, the map $CH_G^*(X_{\leq s}) \rightarrow CH_G^*(X_{\leq t})$ is surjective. We now consider the commutative diagram

$$\begin{array}{ccc} CH_G^*(M_s) & \xrightarrow{j_{\leq s}^*} & CH_G^*(X_{\leq s}) \\ \downarrow & & \downarrow \\ CH_G^*(M_{s,\leq t}) & \xrightarrow{j_{s,\leq t}^*} & CH_G^*(X_{\leq t}). \end{array}$$

Since the composite map in the bottom row of 5.5 is identity, we see by the homotopy invariance that $j_{\leq s}^*$ is surjective. Thus $j_{s,\leq t}^*$ is also surjective. Applying this

surjectivity for $j_{s,\leq t}^*$ and $j_{s,\leq t-1}^*$, we obtain the following commutative diagram

$$(5.8) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & CH_G^*(N_{s,t}) & \xrightarrow{i_{s,t*}} & CH_G^*(M_{s,t}) & \xrightarrow{j_{s,t}^*} & CH_G^*(X_t) \longrightarrow 0 \\ & & \downarrow \eta_{s,t*} & & \downarrow \delta_{s,t*} & & \downarrow f_{t*} \\ 0 & \longrightarrow & CH_G^*(N_{s,\leq t}) & \xrightarrow{i_{s,\leq t*}} & CH_G^*(M_{s,\leq t}) & \xrightarrow{j_{s,\leq t}^*} & CH_G^*(X_{\leq t}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & CH_G^*(N_{s,\leq t-1}) & \xrightarrow{i_{s,\leq t-1*}} & CH_G^*(M_{s,\leq t-1}) & \xrightarrow{j_{s,\leq t-1}^*} & CH_G^*(X_{\leq t-1}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

such that the second and the third rows are exact. All the columns are exact by Corollary 5.7 and Proposition 5.2. We conclude that the localization sequence of the top row is also exact. This proves the surjectivity part of the theorem.

Next we note from 5.2 that $N_{s,\leq t}$ and $N_{s,t}$ are the principal Cartier divisors on $M_{s,\leq t}$ and $M_{s,t}$ respectively. We conclude from Theorem 2.5 that the composites $i_{s,\leq t}^* \circ i_{s,\leq t*}$ and $i_{s,t}^* \circ i_{s,t*}$ are zero. The above diagram now automatically defines the specializations $\overline{Sp}_{X,s}^{\leq t}$ and $\overline{Sp}_{X,s}^t$ and gives the desired factorization of $i_{s,\leq t}^*$ and $i_{s,t}^*$. Since $i_{s,t}^*$ and $j_{s,t}^*$ are ring homomorphisms, and since the latter is surjective as shown in 5.8, we deduce that $\overline{Sp}_{X,s}^t$ is also a ring homomorphism. The map $\overline{Sp}_{X,s}^{\leq t}$ is a ring homomorphism for the same reason.

We are now left with the proof of the commutativity of 5.7. To prove that the right square commutes, we consider the following diagram.

$$(5.9) \quad \begin{array}{ccccc} CH_G^*(M_{s,t}) & \xrightarrow{\delta_{s,t*}} & CH_G^*(M_{s,\leq t}) & & \\ \downarrow i_{s,t}^* & \searrow j_{s,t}^* & \downarrow & \searrow j_{s,\leq t}^* & \\ & CH_G^*(X_t) & \xrightarrow{i_{s,\leq t}^*} & CH_G^*(X_{\leq t}) & \\ & \swarrow \overline{Sp}_{X,s}^t & \downarrow & \swarrow \overline{Sp}_{X,s}^{\leq t} & \\ CH_G^*(N_{s,t}) & \xrightarrow{\eta_{s,t*}} & CH_G^*(N_{s,\leq t}) & & \end{array}$$

It is easy to check that $N_{s,\leq t}$ and $M_{s,t}$ are Tor-independent over $M_{s,\leq t}$ and hence the back face of the above diagram commutes by Lemma 5.9. The upper face commutes by diagram 5.8. Since $j_{s,t}^*$ is surjective, a diagram chase shows that the lower face also commutes, which is what we needed to prove.

Finally, since we have shown that $\eta_{s,t*}$ is injective, and the right square commutes, it now suffices to show that the composite square in 5.7 commutes in order to show that the left square commutes.

By the projection formula, the composite maps $f_{t*} \circ f_t^*$ and $\eta_{s,t*} \circ \eta_{s,t}^*$ are multiplication by $f_{t*}(1)$ and $\eta_{s,t*}(1)$ respectively. Since $\overline{Sp}_{X,s}^{\leq t}$ and $\overline{Sp}_{X,s}^t$ are ring homomorphisms, it suffices to show that

$$\overline{Sp}_{X,s}^{\leq t}(f_{t*} \circ j_{s,t}^*(1)) = \overline{Sp}_{X,s}^{\leq t}(f_{t*}(1)) = \eta_{s,t*}(1).$$

But this follows directly from the commutativity of the right square. \square

Lemma 5.9. *Let G be a linear algebraic group and let*

$$\begin{array}{ccc} W & \xrightarrow{i'} & Y \\ j' \downarrow & & \downarrow j \\ Z & \xrightarrow{i} & X \end{array}$$

be a fiber diagram of closed immersions of G -varieties such that X and Y are smooth. Then one has $i^ \circ j_* = j'_* \circ i'^* : CH_G^*(Y, \cdot) \rightarrow CH_G^*(Z, \cdot)$.*

Proof. By choosing a good pair (V, U) for the G -action and then considering the appropriate mixed quotients, we can reduce to proving the lemma for the non-equivariant higher Chow groups. But this is shown in Lemma 2.3. \square

6. DECOMPOSITION THEOREM FOR EQUIVARIANT HIGHER CHOW GROUPS

We use the specialization maps to prove a decomposition theorem for the equivariant higher Chow groups of $X \in \mathcal{V}_G^S$, where G is a split diagonalizable group. We continue with the notations of the previous section.

Proposition 6.1. *The restriction maps*

$$CH_G^*(X_{\leq s}) \xrightarrow{(f_s^*, g_s^*)} CH_G^*(X_s) \times CH_G^*(X_{< s})$$

define an isomorphism of rings

$$CH_G^*(X_{\leq s}) \xrightarrow{\cong} CH_G^*(X_s) \times_{CH_G^*(N_s^0)} CH_G^*(X_{< s}),$$

where $CH_G^(X_s) \xrightarrow{\eta_{s, \leq s-1}^*} CH_G^*(N_s^0)$ is the pull-back*

$$CH_G^*(X_s) \xrightarrow{\cong} CH_G^*(N_s) \rightarrow CH_G^*(N_s^0)$$

and

$$CH_G^*(X_{< s}) \xrightarrow{\overline{Sp}_{X,s}^{\leq s-1}} CH_G^*(N_{s, \leq s-1}) = CH_G^*(N_s^0)$$

is the specialization map of Theorem 5.8.

Proof. We only need to identify the pull-back and the specialization maps with the appropriate maps of Proposition 5.2. In the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & CH_G^*(X_s) & \xrightarrow{f_{s,\infty}^*} & CH_G^*(N_s) & \xrightarrow{\eta_{s,\leq s-1}^*} & CH_G^*(N_s^0) \longrightarrow 0 \\
 & & & \searrow c_{d_s}^G & \downarrow f_{s,\infty}^* & & \\
 & & & & CH_G^*(X_s) & &
 \end{array}$$

where $f_{s,\infty} : X_s \rightarrow N_s$ is the 0-section embedding, the top sequence is exact, and the lower triangle commutes by Corollary 2.6. Since $f_{s,\infty}^*$ is an isomorphism, this immediately identifies the pull-back map of the proposition with the quotient map $CH_G^*(X_s) \rightarrow \frac{CH_G^*(X_s)}{(c_{d_s}^G(N_s))}$.

Next we consider the following diagram.

$$\begin{array}{ccc}
 CH_G^*(X_{\leq s}) & \xrightarrow{f_{s,\leq s-1}^*} & CH_G^*(X_{< s}) \\
 \overline{Sp}_{X,s}^{\leq s} \downarrow & & \downarrow \overline{Sp}_{X,s}^{\leq s-1} \\
 CH_G^*(N_s) & \longrightarrow & CH_G^*(N_s^0) \\
 f_{s,\infty}^* \downarrow & \nearrow \eta_{s,\leq s-1}^* & \\
 CH_G^*(X_s) & &
 \end{array}$$

Since the top horizontal arrow in the above diagram is surjective, we only need to show that $\overline{Sp}_{X,s}^{\leq s-1} \circ f_{s,\leq s-1}^* = \eta_{s,\leq s-1}^* \circ f_s^*$ in order to identify $\overline{Sp}_{X,s}^{\leq s-1}$ with the map j^* of Proposition 5.2. It is clear from the diagram 5.8 and the definition of the specialization maps that the top square above commutes. We have just shown above that the lower triangle also commutes. This reduces us to showing that

$$(6.1) \quad f_{s,\infty}^* \circ \overline{Sp}_{X,s}^{\leq s} = f_s^*.$$

If $X_s \times \mathbb{P}^1 \xrightarrow{F_s} M_s$ denotes the embedding (cf. 2.3), then for $x \in CH_G^*(X_{\leq s})$, we can write $x = j_{\leq s}^*(y)$ by Theorem 5.8. Then

$$\begin{aligned}
 f_{s,\infty}^* \circ \overline{Sp}_{X,s}^{\leq s} \circ j_{\leq s}^*(y) &= f_{s,\infty}^* \circ i_{s,\leq s}^*(y) = g_{\infty,\leq s}^* \circ F_s^*(y) = g_{0,\leq s}^* \circ F_s^*(y) \\
 &= f_s^* \circ j_{\leq s}^*(y) = f_s^*(x),
 \end{aligned}$$

where the second inequality follows from Lemma 5.9. This proves 6.1 and the proposition. \square

We need the following algebraic result before we prove the main result of this section. Let A be a \mathbb{Q} -algebra (not necessarily commutative). For any linear form $f(\underline{t}) = \sum_{i=1}^n a_i t_i$ in $A[t_1, \dots, t_n]$ such that $a_i \in \mathbb{Q}$ for each i , let $c(f)$ denote the vector $(a_1, \dots, a_n) \in \mathbb{Q}^n$ consisting of the coefficients of the form f .

Lemma 6.2. *Let A be as above and let $S = \{f_1, \dots, f_s\}$ be a set of linear forms in $A[t_1, \dots, t_n]$ such that the vectors $\{c(f_1), \dots, c(f_s)\}$ are linearly independent in \mathbb{Q}^n . Let*

$$\gamma_j = \sum_{i=0}^{d_j} m_i^j f_j^i$$

such that $m_{d_j}^j \in \mathbb{Q}^$ for $1 \leq j \leq s$, and $m_{j'}^j \in Z(A)$ for all j, j' . Then one has*

$$(\gamma_1 \cdots \gamma_s) = \bigcap_{j=1}^s (\gamma_j)$$

as ideals in $A[t_1, \dots, t_n]$.

Proof. Using a simple induction, it suffices to show that for $j \neq j'$, the relation $\gamma_j | q \gamma_{j'}$ implies that $\gamma_j | q$. So we can assume $S = \{f_1, f_2\}$. Extend $\{c(f_1), c(f_2)\}$ to a basis B of \mathbb{Q}^n . Applying the linear automorphism of $A[t_1, \dots, t_n]$ given by the invertible matrix B , we can assume that $f_j = t_j$ for $j = 1, 2$. Now the proof follows along the same lines as the proof of Lemma 4.9 of [34]. We skip the details. \square

Theorem 6.3. *Let G be a split diagonalizable group of dimension n and let $X \in \mathcal{V}_G^S$. The ring homomorphism*

$$CH_G^*(X) \longrightarrow \prod_{s=0}^n CH_G^*(X_s)$$

is injective. Moreover, its image consists of the n -tuples (α_s) in the product with the property that for each $s = 1, \dots, n$, the pull-back of $\alpha_s \in CH_G^(X_s)$ in $CH_G^*(N_{s,s-1})$ is same as $\overline{Sp}_{X,s}^{s-1}(\alpha_{s-1}) \in CH_G^*(N_{s,s-1})$. In other words, there is a ring isomorphism*

$$CH_G^*(X) \xrightarrow{\cong} CH_G^*(X_n) \times_{CH_G^*(N_{n,n-1})} CH_G^*(X_{n-1}) \times_{CH_G^*(N_{n-1,n-2})} \cdots \times_{CH_G^*(N_{1,0})} CH_G^*(X_0).$$

Proof. We prove by the induction on the largest integer s such that $X_s \neq \emptyset$.

If $s = 0$, there is nothing to prove. If $s > 0$, we have by induction

$$(6.2) \quad CH_G^*(X_{<s}) \xrightarrow{\cong} CH_G^*(X_{s-1}) \times_{CH_G^*(N_{s-1,s-2})} \cdots \times_{CH_G^*(N_{1,0})} CH_G^*(X_0).$$

Using this and Proposition 6.1, it suffices to show that if $\alpha_s \in CH_G^*(X_s)$ and if $\alpha_{<s} \in CH_G^*(X_{<s})$ with the restriction $\alpha_{s-1} \in CH_G^*(X_{s-1})$ are such that $\alpha_s \mapsto \alpha_s^0 \in CH_G^*(N_s^0)$ and $\alpha_s \mapsto \alpha_{s,s-1} \in CH_G^*(N_{s-1,s-2})$, then

$$\overline{Sp}_{X,s}^{<s-1}(\alpha_{<s}) = \alpha_s^0 \text{ iff } \overline{Sp}_{X,s}^{s-1}(\alpha_{s-1}) = \alpha_{s,s-1}.$$

Using the commutativity of the left square in Theorem 5.8, this is reduced to showing that the restriction map $CH_G^*(N_s^0) \rightarrow CH_G^*(N_{s-1,s-2})$ is injective.

To prove this, we first use Proposition 5.5 to assume that the toral component T of the isotropy groups of the points of X_s is fixed, and choose a splitting $G = D \times T$.

Now, following the proof of the analogous result for K -theory (cf. [34, Theorem 4.5]), we can write

$$N_s = E = \bigoplus_{i=1}^q E_i \quad \text{and} \quad N_{s,s-1} = \prod_i E_i^0,$$

where each E_i is of the form $\bigoplus E_{m_j \chi_i} \otimes m_j \chi_i$, such that for $i \neq j$, χ_i and χ_j are linearly independent characters of T , and E_i^0 is embedded in E by setting all the other components equal to zero. Let $d_i = \text{rank}(E_i)$.

Now we see from Proposition 6.1 that

$$\text{Ker}(CH_G^*(X_s) \rightarrow CH_G^*(N_{s,s-1})) = \bigcap_i (c_{d_i}^G(E_i)) \quad \text{and}$$

$$\text{Ker}(CH_G^*(X_s) \rightarrow CH_G^*(N_s^0)) = (c_{d_s}^G(N_s)) \quad \text{with} \quad d_s = \sum d_i.$$

Putting $\gamma_i = c_{d_i}^G(E_i)$ and $\gamma = c_{d_s}^G(N_s)$, we are then reduced to showing that

$$(6.3) \quad (\gamma) = \left(\prod_i \gamma_i \right) = \bigcap_i (\gamma_i)$$

in $CH_D^*(X_s)[t_1, \dots, t_s]$.

However, we have seen in the proof of Proposition 5.4 that each γ_i is of the form

$$\gamma_i = u_i^{d_i} + \alpha_{d_i-1}^i u_i^{d_i-1} + \dots + \alpha_1^i u_i + \alpha_0^i,$$

where $\alpha_j^i \in CH_D^*(X_s, 0) \subset Z(CH_D^*(X_s))$ and $u_i = c_1^T(L_{\chi_i}) = \sum_{j=1}^s b_j^i t_j \neq 0$ in $\mathbb{Q}[t_1, \dots, t_s]$. Moreover, the linear independence of χ_i 's implies that the vectors $\{c(u_1), \dots, c(u_q)\}$ are linearly independent. We now apply Lemma 6.2 to conclude the proof of 6.3 and hence the theorem. \square

7. EQUIVARIANT CHERN CHARACTER AND RIEMANN-ROCH MAPS

The equivariant Riemann-Roch map for the Grothendieck group of equivariant coherent sheaves on a variety has been constructed by Edidin and Graham in [6]. In this section, we construct these maps for the higher equivariant K -theory and study their properties. Following the techniques of Gillet (cf. [12]) for the construction of the Riemann-Roch maps, we first define the equivariant Chern character map for the smooth varieties, and then use this Chern character to define the Riemann-Roch for all varieties. Recall from Section 1 that for any G -variety X , $\widehat{G_i^G(X)}$ denotes the I_G -adic completion of the $R(G)$ -module $G_i^G(X)$, where I_G is the ideal of virtual representations of G of rank zero in the representation ring $R(G)$. We shall follow the notations of Section 4 for the various completions of $R(G)$ and $S(G)$ -modules in the rest of this paper. In particular, $\widehat{K_i^G(X)}$ will denote the weak completion $K_i^G(X) \otimes_{R(G)} \widehat{R(G)}$. For a map $f : Y \rightarrow X$ in \mathcal{V}_G , we make the

convention in this paper that the induced pull-back (or push-forward) map on K -theory will be denoted by f^* (f_*), and the map on the higher Chow groups will be denoted by \bar{f}^* (\bar{f}_*).

Proposition 7.1. *Let G be a linear algebraic group and let $X \in \mathcal{V}_G^S$. Then for every $i \geq 0$, there is a Chern character map*

$$ch_X^G : K_i^G(X) \longrightarrow \prod_j CH_G^j(X, i) = \overline{CH_G^*(X, i)}$$

with the following properties.

- (i) ch^G is a contravariant functor from \mathcal{V}_G^S to $\text{Vec}_{\mathbb{Q}}$.
- (ii) For $\alpha \in K_0^G(X)$ and $x \in K_i^G(X)$, one has $ch_X^G(\alpha x) = ch_X^G(\alpha) \cdot ch_X^G(x)$, where the $\overline{CH_G^*(X, 0)}$ -module structure on $\overline{CH_G^*(X, i)}$ is induced by the intersection product.
- (iii) ch_X^G factors through the I_G -adic completion

$$\widehat{K_i^G(X)} \xrightarrow{\widehat{ch_X^G}} \overline{CH_G^*(X, i)}.$$

- (iv) If G acts freely on X , then ch_X^G coincides with the non-equivariant Chern character map $ch_{X/G}$ under the identifications $K_i^G(X) = K_i(X/G)$ and $\overline{CH_G^*(X, i)} = CH^*(X/G, i)$ (cf. Proposition 2.2).

Proof. In order to define ch_X^G , it suffices to define the component $(ch_X^G(x))_j$ of $ch_X^G(x)$ in $CH_G^j(X, i)$ for every $j \geq 0$, whenever $x \in K_i^G(X)$. So we fix $j \geq 0$ and choose a good pair (V, U) for G corresponding to j . Let $\pi_{UX} : X \times U \rightarrow X$ be the projection map, which is flat. We define $(ch_X^G(x))_j$ to be the j th component of the composite map

$$K_X^G(X) \xrightarrow{\pi_{UX}^*} K_{X \times U}^G(X \times U) \xrightarrow{\cong} K_i(X_G) \xrightarrow{ch_{X_G}} CH^*(X_G, i),$$

where $ch_{X_G} : K_i(X_G) \rightarrow CH^*(X_G, i)$ is the non-equivariant Chern character map of Bloch and Gillet (cf. [12, Definition 2.34], [2, Section 7]). The proof of the independence of the above definition of the choice of the good pair (V, U) is same as the proof of Proposition 3.1 in [6], using the fact that the non-equivariant K -theory and higher Chow groups satisfy the homotopy invariance property.

To prove the contravariance property of ch^G , we can again reduce to the non-equivariant case as above, where this is already known (cf. [12], see also [10, Theorem 1.10]). The same argument also proves (ii) as the non-equivariant Chern character is known to be a ring isomorphism (*loc. cit.*).

To prove (iii), we have $ch_k^G(I_G) \subset \prod_{j \geq 1} CH_G^j(k, 0)$ by [6, 3.1], and hence $ch_X^G(I_G^n K_i^G(X)) \subset \prod_{j \geq n} CH_G^j(X, i)$ by (ii). This gives a map of inverse systems

$$(7.1) \quad \frac{K_i^G(X)}{I_G^n K_i^G(X)} \xrightarrow{ch_X^G} \frac{\prod_j CH_G^j(X, i)}{\prod_{j \geq n} CH_G^j(X, i)}.$$

Taking the inverse limits both sides and using 4.1, we get the required map in (iii). The last assertion follows directly from the construction of ch_X^G above and by applying the contravariance property of the non-equivariant Chern character for the natural map $X_G \rightarrow X/G$. \square

For a G -variety X which is not necessarily smooth, we define (cf. [7, 2.6]) the equivariant operational Chow groups $A_G^j(X)$ as operations $c(Y \rightarrow X) : CH_G^{j'}(Y, \cdot) \rightarrow CH_G^{j+j'}(Y, \cdot)$ for any map $Y \rightarrow X$ in \mathcal{V}_G , satisfying the same properties as in the non-equivariant case. If E is an equivariant vector bundle of rank r on X , then for any map $Y \xrightarrow{f} X$, we can apply Proposition 2.2(v), to get the Chern class operations $c_j^G(f^*(E)) : CH_G^{j'}(Y, \cdot) \rightarrow CH_G^{j+j'}(Y, \cdot)$ and hence elements in $A_G^j(X)$. This defines the Chern character

$$(7.2) \quad \overline{ch}_X^G : K_0^G(X) \longrightarrow \prod_{j=0}^{\infty} A_G^j(X)$$

which is a ring homomorphism and the Todd class

$$Td_X^G(E) = \prod_{j=1}^r \frac{x_j}{1 - e^{-x_j}}$$

where $\{x_1, \dots, x_r\}$ are the Chern roots of E (cf. [6, 3.1]). The Todd class of E is an invertible element of $\prod_{j=0}^{\infty} A_G^j(X)$. Let \mathfrak{m}_X denote the maximal ideal of the ring $K_0^G(X)$ consisting of the isomorphism classes of the vector bundles of virtual rank zero. The Chern character \overline{ch}_X^G has the following important property that will be useful to us.

Proposition 7.2. *Let G be a linear algebraic group and let $X \in \mathcal{V}_G$. Then the map \overline{ch}_X^G factors through the localization*

$$K_0^G(X)_{\mathfrak{m}_X} \xrightarrow{\overline{ch}_X^G} \prod_{j=0}^{\infty} A_G^j(X).$$

Proof. If X is smooth, this follows from [6, Theorem 4.1, Theorem 6.1]. Now suppose that X is not necessarily smooth. We need to show that for any element $\alpha \in K_0^G(X)$ which is not in \mathfrak{m}_X , the image of $\overline{ch}_X^G(\alpha)$ is an invertible element

of $\prod_{j=0}^{\infty} A_G^j(X)$. Before we show this, we observe that $K_0^G(X)$ has a canonical decomposition

$$(7.3) \quad K_0^G(X) = \mathbb{Q} \oplus \mathfrak{m}_X$$

as a \mathbb{Q} -vector space. Given any $\alpha \in K_0^G(X)$, we can use Lemma 7.3 below to get a G -equivariant embedding $X \xrightarrow{i} M$ such that M is smooth and $\alpha = i^*(\beta)$ for some $\beta \in K_0^G(M)$. Now the contravariance property of \overline{ch}^G gives a commutative diagram

$$\begin{array}{ccc} K_0^G(M) & \xrightarrow{\overline{ch}_M^G} & \prod_{j=0}^{\infty} A_G^j(M) \\ i^* \downarrow & & \downarrow \bar{i}^* \\ K_0^G(X) & \xrightarrow{\overline{ch}_X^G} & \prod_{j=0}^{\infty} A_G^j(X) \end{array}$$

Since i^* is a ring homomorphism which preserves the rank, we see from 7.3 that if $\alpha \notin \mathfrak{m}_X$, then β is also not in \mathfrak{m}_M and hence $\overline{ch}_M^G(\beta)$ is a unit in $\prod_{j=0}^{\infty} A_G^j(M)$ by the smooth case. Hence $\overline{ch}^G(\alpha) = \bar{i}^* \circ \overline{ch}_M^G(\beta)$ is a unit in $\prod_{j=0}^{\infty} A_G^j(X)$. \square

Lemma 7.3. *For any $X \in \mathcal{V}_G$ and $\alpha \in K_0^G(X)$, there is an equivariant closed embedding $X \xrightarrow{i} M$ with $M \in \mathcal{V}_G^S$ and a class $\beta \in K_0^G(M)$ such that $\alpha = i^*(\beta)$.*

Proof. Since $K_0^G(X)$ is the group of isomorphism classes of equivariant vector bundles, we can reduce the problem to α being a finite collection of vector bundles. Then by the diagonal embedding of X into a product of smooth varieties, we reduce the problem to the case when α is an equivariant vector bundle E of rank r .

Since G acts linearly on X , there is an equivariant closed embedding $X \xrightarrow{f} Y$ with Y smooth. Put $E' = f_*(E)$. Then by [30], there is a G -equivariant vector bundle F on Y and an equivariant surjection $F \twoheadrightarrow E'$. Let M be the Grassman bundle of rank r quotient bundles of F on Y and let $M \xrightarrow{p} Y$ be the projection map. Let Q denote the universal quotient bundle of rank r on M . It is now easy to see that the action of G on Y and F induces a natural G -action on M such that p is a G -equivariant map and Q is a G -equivariant bundle on M . The universal property of the Grassman bundle then implies that the map f factors through a map $X \xrightarrow{i} M$ such that $E = i^*(Q)$. Clearly, M is smooth and i is a closed embedding. \square

The following property of the non-equivariant Riemann-Roch map $\tau_X : G_i(X) \rightarrow CH^*(X, i)$ (cf. [2, 7.4]) will be used frequently in the construction of these maps in the equivariant setting.

Lemma 7.4. *Let $f : Y \rightarrow X$ be a morphism of varieties such that either*

- (i) *f is a vector bundle morphism X , or*
- (ii) *X and Y are smooth and f is an l.c.i. morphism.*

Then one has $\tau_Y \circ f^ = Td_Y(T_f) \cdot (\bar{f}^* \circ \tau_X)$.*

Proof. First suppose X is not necessarily smooth and Y is a vector bundle over X . Choose a finite open covering \mathcal{U} of X such that the restriction of f on every $U \in \mathcal{U}$ is trivial. Since the Riemann-Roch map commutes with the restriction to open covers (cf. [12, Theorem 4.1]), we get a commutative diagram

$$\begin{array}{ccc} G_i(X) & \longrightarrow & \check{H}(\mathcal{U}, \mathcal{K}) \\ \tau_X \downarrow & & \downarrow \tau_{\mathcal{U}} \\ CH^*(X, i) & \longrightarrow & \check{H}(\mathcal{U}, \mathcal{CH}(-, i)), \end{array}$$

where the terms on the right are the Čech cohomology of the sheaves of K -groups and Chow groups for the cover \mathcal{U} . The horizontal maps are isomorphisms by the Mayer-Vietoris property of K -theory and higher Chow groups. Thus it suffices to prove the desired result for the open subsets in \mathcal{U} . Hence we can assume that f is a trivial bundle, which can be further assumed to be of rank one by induction.

Thus we have $Y = X \times \mathbb{A}^1$ and f is the projection map. Put $\tilde{Y} = X \times \mathbb{P}^1$ and let $\tilde{f} : \tilde{Y} \rightarrow X$ be the projection and $j : Y \rightarrow \tilde{Y}$ the inclusion. Now we apply the isomorphisms $G_i(Y) = G_i(X) \otimes G_0(\mathbb{P}^1)$, $CH^*(Y, i) = CH^*(X, i) \otimes CH^*(\mathbb{P}^1, 0)$ and the proof of the Riemann-Roch theorem for the map $\mathbb{P}_X^1 \rightarrow X$ (cf. [12, Lemma 4.4]) to get

$$\begin{aligned} \tau_{\tilde{Y}} \circ \tilde{f}^*(a) &= \tau_{\tilde{Y}}(a \otimes 1) &= \tau_X(a) \otimes \tau_{\mathbb{P}^1}(1) \\ &= \tau_X(a) \otimes Td(\mathbb{P}^1) &= (1 \otimes Td(\mathbb{P}^1)) \cdot (\tau_X(a) \otimes 1) \\ &= Td(T_{\tilde{f}}) \cdot (\tau_X(a) \otimes 1) &= Td(T_{\tilde{f}}) \cdot (\widetilde{f}^* \circ \tau_X(a)). \end{aligned}$$

Finally, we have

$$\tau_Y \circ f^* = \tau_Y \circ j^* \circ \tilde{f}^* = \bar{j}^* \circ \tau_{\tilde{Y}} \circ \tilde{f}^* = \bar{j}^* \circ \widetilde{f}^* \circ \tau_X = \bar{f}^* \circ \tau_X.$$

This proves the part (i).

Now suppose X and Y are smooth and f is an l.c.i. morphism. Then we have

$$\begin{aligned} \tau_Y \circ f^* &= (ch_Y \circ f^*) \cdot Td(Y) &= (\bar{f}^* \circ ch_X) \cdot Td(Y) \\ &= (\bar{f}^* \circ ch_X) \cdot [Td(f^*(X)) \cdot Td(T_f)] &= \bar{f}^*(ch_X \cdot Td(X)) \cdot Td(T_f) \\ &= (\bar{f}^* \circ \tau_X) \cdot Td(T_f) \end{aligned}$$

□

Theorem 7.5. *Let G be a linear algebraic group and let $X \in \mathcal{V}_G$. Then there is a Riemann-Roch map*

$$(7.4) \quad \tau_X^G : G_i^G(X) \rightarrow \overline{CH}_G^*(X, i)$$

which satisfies the following properties.

- (i) τ^G is covariant for proper maps in \mathcal{V}_G .

(ii) For $\alpha \in K_0^G(X)$ and $x \in G_i^G(X)$, one has $\tau_X^G(\alpha x) = \overline{ch}_X^G(\alpha) \cdot \tau_X^G(x)$, with respect to the $K_0^G(X)$ -module structure on $G_i^G(X)$.

(iii) τ_X^G factors through the I_G -adic completion

$$(7.5) \quad \widehat{G_i^G(X)} \xrightarrow{\widehat{\tau_X^G}} \overline{CH_G^*(X, i)}.$$

(iv) For a morphism $f : Y \rightarrow X$ in \mathcal{V}_G such that f is either an equivariant vector bundle morphism, or it is an equivariant l.c.i. morphism in \mathcal{V}_G^S , one has $\tau_Y^G \circ f^* = Td_Y^G(T_f) \cdot (\bar{f}^* \circ \tau_X^G)$. If $j : U \rightarrow X$ is a G -equivariant open immersion, then $\tau_U^G \circ j^* = \bar{j}^* \circ \tau_X^G$.

(v) If G acts freely on X , then τ_X^G coincides with the non-equivariant Riemann-Roch map $\tau_{X/G}$ under the identifications $G_i^G(X) = G_i(X/G)$ and $CH_G^*(X, i) = CH^*(X/G, i)$.

Proof. As in the case of Chern character, we need to define each component of τ_X^G in the product $\prod_{j \geq 0} CH_G^j(X, i)$. So we fix $x \in G_i^G(X)$ and $j \geq 0$ and choose a good pair (V, U) for G corresponding to j . Let $\pi_{UX} : X \times U \rightarrow X$ be the projection map, and consider the diagram

$$\begin{array}{ccc} G_X^G(X) & \xrightarrow{\pi_{UX}^*} G_{X \times U}^G(X \times U) & \xrightarrow{\cong} G_i(X_G) \\ \downarrow & & \downarrow \tau_{X_G} \\ \overline{CH_G^*(X, i)} & \xleftarrow{\quad} & CH^*(X_G, i) \end{array}$$

where $\tau_{X_G} : G_i(X/G) \rightarrow CH^*(X_G, i)$ is the non-equivariant Riemann-Roch map (cf. [2, 7.4]) of Bloch and Gillet. We define

$$(7.6) \quad \tau_X^G(x) = \frac{\tau_{X_G} \circ \pi_{UX}^*(x)}{Td_{X_G}(E_V)},$$

where E_V is the vector bundle $X \times^G (U \times V) \rightarrow X \times^G U = X_G$.

The proof that the above definition is independent of the choice of the good pair (V, U) follows exactly along the lines of the proof of [6, Proposition 31.1]. The only extra ingredient we need in our case is Lemma 7.4.

The properties (ii) and (iii) follow directly from the definition of τ^G and the analogous properties of the non-equivariant Riemann-Roch map (cf. [12, Theorem 4.1]). The proof of property (iii) is same as the proof of the corresponding property of the equivariant Chern character map in Proposition 7.1. The property (iv) follows from the corresponding non-equivariant result in Lemma 7.4. The statement about open immersion follows from the analogous property of the non-equivariant Riemann-Roch (cf. [12, Theorem 4.1]). The proof of property (v) is the same as the proof of Proposition 7.1 (iv). \square

Corollary 7.6. *The Riemann-Roch map factors through the localization*

$$\tau_X^G : G_i^G(X)_{\mathfrak{m}_X} \rightarrow \overline{CH_G^*(X, i)}.$$

Proof. We have seen that \overline{ch}_X^G is a ring homomorphism. Moreover, it follows from Theorem 7.5 (ii) that τ_X^G is $K_0^G(X)$ -linear. Thus it suffices to show that any element $\alpha \in K_0^G(X)$ which is not in \mathfrak{m}_X , acts as a unit in $\overline{CH}_G^*(X, i)$. But this follows immediately from Proposition 7.2. \square

8. RIEMANN-ROCH FOR FINITE GROUP ACTIONS

In this section, we review the equivariant K -theory for finite group actions and prove our Riemann-Roch isomorphism in this case. As remarked before, such an isomorphism can also be obtained using the study of equivariant K -theory for finite group actions in [37]. Let G be a finite group and let X be an equidimensional G -variety. Let $p : X \rightarrow Y = X/G$ be the quotient for the G -action. Note that this quotient always exists and is quasi-projective because we are dealing with the linear actions on quasi-projective varieties. Let $\mathcal{Z}^j(X, \cdot)$ denote the cycle complex of codimension j cycles on X . Then G acts on $\mathcal{Z}^j(X, \cdot)$ by letting it act on an irreducible cycle $\sigma \in \mathcal{Z}^j(X, n)$ by

$$(g, \sigma) \mapsto \mu_g^*(\sigma)$$

and extending this linearly to all of $\sigma \in \mathcal{Z}^j(X, n)$. Note that G here acts on $X \times \Delta^n$ via the diagonal action for the trivial action of Δ^n . In particular, the action of G on $\mathcal{Z}^j(X, \cdot)$ commutes with the boundary maps, and we get an action of G on the complex $\mathcal{Z}^j(X, \cdot)$. Let $\mathcal{Z}_G^j(X, \cdot)$ denote the subcomplex of invariant cycles, and let d^G denote the restriction of the differential d of the complex $\mathcal{Z}^j(X, \cdot)$ to $\mathcal{Z}_G^j(X, \cdot)$. Put

$$\widetilde{CH}_G^j(X, i) = H_i(\mathcal{Z}_G^j(X, \cdot)) \quad \text{for } i \geq 0.$$

This gives a natural map

$$(8.1) \quad \widetilde{CH}_G^j(X, \cdot) \xrightarrow{\delta_X^G} (CH^j(X, \cdot))^G.$$

Lemma 8.1. *There are canonical isomorphisms*

$$CH^j(Y, \cdot) \xrightarrow{p^*} \widetilde{CH}_G^j(X, \cdot) \xrightarrow{\delta_X^G} (CH^j(X, \cdot))^G.$$

Proof. Let $V \subset Y \times \Delta^n$ be an irreducible cycle intersecting all faces of $Y \times \Delta^n$ properly and let $\overline{V} = p^{-1}(V)$. Then $\overline{V} \rightarrow V$ is a finite morphism. Moreover, the trivial action of G on Δ^n implies that $p^{-1}(V \cap Y \times \Delta^m) = \overline{V} \cap X \times \Delta^m$ for all $m \leq n$. Hence H defines a cycle on $X \times \Delta^n$ which is clearly G -invariant. This gives a natural map $\mathcal{Z}^j(Y, \cdot) \xrightarrow{p^*} \mathcal{Z}_G^j(X, \cdot)$. To show that p^* is an isomorphism of complexes, we need to give a unique representation for every cycle $\sigma \in \mathcal{Z}_G^j(X, \cdot)$ in terms of image of p^* . Now we can write σ as

$$\sigma = \sum a_j V_j - \sum a'_j V'_j, \quad \text{with } a_j, a'_j > 0.$$

Then it is easy to see that both the sums on the right are G -invariant. Hence can assume that $\sigma = \sum a_j V_j$ with $a_j > 0$. Since V_j 's are irreducible, σ is G -invariant if and only if it is of the form $\sum b_j H_j$, where each H_j is of the form $\sum_{g \in G} g \widetilde{H}_j$ where

\widetilde{H}_j is an irreducible cycle. Taking $W_j = p(\widetilde{H}_j)$, it is then easy to see that $\sigma = \sum a_j p^{-1}(W_j) = p^{-1}(\delta)$, where W_j 's (and hence δ) are uniquely determined by σ .

We now show that δ_X^G is an isomorphism. The injectivity is proved by defining the trace map $\mathcal{Z}^j(X, \cdot) \xrightarrow{tr} \mathcal{Z}_G^j(X, \cdot)$

$$(8.2) \quad tr(x) = \frac{1}{|G|} \sum_{g \in G} \mu_g^*(x),$$

and checking that $tr \circ \delta_X^G$ is the identity map on the homology. Thus we need to prove the surjectivity of δ_X^G . So let $x \in \mathcal{Z}^j(X, n)$ be such that $d_n(x) = 0$ and $x - gx \in \text{Image}(d_{n+1}) \forall g \in G$ (since $x \in (CH^j(X, n))^G$). Putting $x_g = x - gx$, we get

$$x = \frac{1}{|G|} \sum_{g \in G} x = \frac{1}{|G|} \sum_{g \in G} gx + \frac{1}{|G|} \sum_{g \in G} x_g = tr(x) + y,$$

where $y = \frac{1}{|G|} \sum_{g \in G} x_g \in \text{Image}(d_{n+1})$. Since $d_n(x) = 0$, it is easy to check that $d_n^G(tr(x)) = 0$ and $x = \delta_X^G(tr(x))$. \square

Corollary 8.2. *There are canonical isomorphisms*

$$(8.3) \quad CH_G^*(X, \cdot) \rightarrow (CH^j(X, \cdot))^G \xleftarrow{p^*} CH^*(Y, \cdot).$$

In particular, the natural maps

$$(8.4) \quad CH_G^*(X, \cdot) \rightarrow \widehat{CH_G^*(X, \cdot)} \rightarrow \widehat{CH_G^*(X, \cdot)} \rightarrow \overline{CH_G^*(X, \cdot)}$$

are all isomorphisms.

Proof. To prove 8.3, we only need to prove the first isomorphism. So fix $j \geq 0$ and choose a good pair (V, U) for the G -action corresponding to j . Then we get the canonical isomorphisms

$$CH_G^j(X, \cdot) \xrightarrow{\cong} CH^j(X_G, \cdot) \xrightarrow{\cong} (CH^j(X \times U, \cdot))^G,$$

where the second map is isomorphism by Lemma 8.1. On the other hand we have natural G -equivariant maps

$$CH^j(X, \cdot) \rightarrow CH^j(X \times V, \cdot) \rightarrow CH^j(X \times U, \cdot).$$

The first map is an isomorphism by the homotopy invariance and the second map is isomorphism by the localization sequence since (V, U) is a good pair. Taking the G -invariants both sides, we conclude the proof.

The isomorphism of the last and the composite of all maps in 8.4 follows from the isomorphisms in 8.3, which implies that the J_G -adic filtration and the filtration by grading on $CH_G^j(X, \cdot)$ are nilpotent. The isomorphism of the first map (hence the second map) now follows from Corollary 4.8. \square

Corollary 8.3. *Let G be a linear algebraic group over k such that all its irreducible components are also defined over k . Let G^0 denote the identity component of G and let $H = G/G^0$. Let X be a quasi-projective k -variety with a G -action. Then the finite group H naturally acts on $CH_{G^0}^*(X, i)$ such that $(CH_{G^0}^*(X, i))^H \cong CH_G^*(X, i)$.*

Proof. We first observe that since all the components of G are defined over k , H is a finite constant group. Fix $j \geq 0$ and choose a good pair (V, U) for the G -action on X . Then this is also a good pair for the G^0 -action on X . Moreover, H naturally acts on the mixed quotient $X \overset{G^0}{\times} U$ such that the corresponding quotient is $X \overset{G}{\times} U$. Hence by Lemma 8.1, H acts on $CH^j(X \overset{G^0}{\times} U, i)$ such that $\left(CH^j(X \overset{G^0}{\times} U, i)\right)^H \cong CH^j(X \overset{G}{\times} U, i)$. The corollary now follows from the isomorphisms $CH_{G^0}^j(X, i) \cong CH^j(X \overset{G^0}{\times} U, i)$ and $CH_G^j(X, i) \cong CH^j(X \overset{G}{\times} U, i)$. \square

For a finite group G acting on X , let $\sigma : G \times X \rightarrow X$ denote the action map. It is then clear that σ is finite and étale. Put $\alpha_X = \sigma_*([\mathcal{O}_{G \times X}]) \in K_0^G(X)$.

Lemma 8.4. *For G and X as above, one has $\tau_{G \times X}^G \circ \sigma^* = \bar{\sigma}^* \circ \tau_X^G$.*

Proof. We embed X equivariantly into a smooth G -variety M . By Corollary 8.3, we can choose a good pair (V, U) for the G -action such that $CH_G^*(X, \cdot) = CH^*(X_G, \cdot)$ and same for M . Then as in the proof of Theorem 7.5 (iv), we are reduced to proving the non-equivariant version of the lemma for the induced map of quotients $G \overset{G}{\times} (X \times U) \xrightarrow{f} X \overset{G}{\times} U$ which is also finite and étale. Note that $G \overset{G}{\times} (X \times U) \cong X \times U$ and the map f is the quotient map for the free action of G on $X \times U$.

Letting $M \times U \xrightarrow{q} M \overset{G}{\times} U$ denote the corresponding map for M , our problem is reduced to proving that for a fiber diagram

$$(8.5) \quad \begin{array}{ccc} Y & \xrightarrow{i} & Z \\ p \downarrow & & \downarrow q \\ S & \xrightarrow{j} & W \end{array}$$

of varieties such that the bottom arrow is the closed embedding of quotients for a free action of a finite group G on the top arrow, which is an equivariant closed embedding, one has $\tau_Y \circ p^* = \bar{p}^* \circ \tau_S$.

Before we prove this, we recall the construction of the Riemann-Roch map from [2]. Given any variety X , we first embed $X \xrightarrow{i} M$ with M smooth. Let U denote

the complement of X in M . Then there is a diagram of homotopy fibration of spectra

$$(8.6) \quad \begin{array}{ccccc} G(X) & \xrightarrow{i_*} & G(M) & \xrightarrow{j^*} & G(U) \\ & & \downarrow ch_M & & \downarrow ch_U \\ \mathcal{H}_X & \xrightarrow{i_*} & \mathcal{H}_M & \xrightarrow{j^*} & \mathcal{H}_U, \end{array}$$

where the terms on the bottom arrow are spectra whose homotopy groups give the higher Chow groups (cf. [10, Theorem 1.10]). Since this is a diagram of homotopy fibrations and the right square commutes, we get an induced Chern character with support $ch_X^M : G(X) \rightarrow \mathcal{H}_X$. Then one defines the Riemann-Roch map for X on the homotopy groups by $\tau_X(x) = Td(i^*T_M) \cdot ch_X^M(x)$. This is independent of M .

Coming back to the proof of 8.5, we first claim that $p_* \circ p^* = |G|id$. But the projection formula reduces this to showing that $p_*(1) = |G|$. By the Riemann-Roch theorem (cf. [2]), it suffices to show that $\bar{p}_* \circ \tau_Y(1) = |G|\tau_S(1)$. Using the above construction of the Riemann-Roch map, we have

$$\begin{aligned} \bar{p}_* \circ \tau_Y(1) &= \bar{p}_* [ch_Y(1)\bar{i}^*(Td(Z))] \\ &= \bar{p}_* \circ \bar{i}^*(Td(Z)) \\ &= \bar{p}_* \circ \bar{i}^* \circ \bar{q}^*(Td(W)) \\ &= \bar{p}_* \circ \bar{p}^* \circ \bar{j}^*(Td(W)) \\ &= |G|\tau_S(1), \end{aligned}$$

where the third equality holds because q is a finite and étale map of smooth varieties. This proves the claim.

Now for any $a \in G_i(S)$, we have

$$\begin{aligned} \bar{p}^* \circ \tau_S(a) &= \bar{p}^* \circ \tau_S \left(\frac{1}{|G|} p_* \circ p^*(a) \right) \\ &= \frac{1}{|G|} (\bar{p}^* \circ \tau_S \circ p_* \circ p^*(a)) \\ &= \frac{1}{|G|} (\bar{p}^* \circ \bar{p}_* \circ \tau_Y \circ p^*(a)) \\ &= tr(\tau_Y \circ p^*(a)) \\ &= \tau_Y \circ p^*(tr(a)) \\ &= \tau_Y \circ p^*(a). \end{aligned}$$

Here, the first equality follows from the above claim. The fourth equality occurs because of the fact that for any irreducible cycle $V \in \mathcal{Z}^*(Y, n)$, one has $\bar{p}^* \circ \bar{p}_*(V) = \sum_{g \in G} \mu^*(V) = |G|tr(V)$. Finally, the last equality holds because G acts trivially on S . This completes the proof of the lemma. \square

Theorem 8.5. *For $X \in \mathcal{V}_G$ as above, the Riemann-Roch map $G_i^G(X) \xrightarrow{\tau_X^G} CH_G^*(X, i)$ is surjective. Moreover, for any $x \in G_i^G(X)$, one has $\tau_X^G(x) = 0$ if and only if $\alpha_X \cdot x = 0$.*

Proof. Before we begin the proof, we note that τ_X^G maps $G_i^G(X)$ into $\overline{CH_G^*(X, i)}$, but we can now replace the latter by $CH_G^*(X, i)$ using Corollary 8.3.

To show the surjectivity of τ_X^G , we consider the following diagram

$$\begin{array}{ccc} G_i^G(G \times X) & \xrightarrow{\tau_{G \times X}^G} & CH_G^*(G \times X, i) \\ \sigma_* \downarrow & & \downarrow \bar{\sigma}_* \\ G_i^G(X) & \xrightarrow{\tau_X^G} & CH_G^*(X, i) \end{array}$$

which commutes by Theorem 7.5. Since G acts freely on $G \times X$, we can apply Theorem 7.5 and the non-equivariant Riemann-Roch (cf. [2, Theorem 9.1]) to conclude that the top horizontal arrow is an isomorphism. Moreover, as σ is finite and étale, we have $\bar{\sigma}_* \circ \bar{\sigma}^* = |G|id$. In particular, $\bar{\sigma}_*$ is surjective and hence so is τ_X^G .

To prove the remaining part of the theorem, we first show that if G acts freely on X , then α_X is an invertible element of $K_0^G(X)$. Let $X \xrightarrow{p} Y = X/G$ be the quotient map. Then $K_0^G(X) \cong K_0(Y)$ and $\alpha_X = p_*([\mathcal{O}_X])$. Now the assertion follows from the fact that $K_0(Y)$ has a canonical decomposition $K_0(Y) = \widetilde{K_0(Y)} \oplus \mathbb{Q}$, where $\widetilde{K_0(Y)}$ is nilpotent and α_X has positive rank.

Now suppose $x \in G_i^G(X)$ is such that $\tau_X^G(x) = 0$. Then by Lemma 8.4, we get $\tau_{G \times X}^G \circ \sigma^*(x) = 0$, which in turn implies that $\sigma^*(x) = 0$ as $\tau_{G \times X}^G$ is an isomorphism. Composing this with σ_* and applying the projection formula, we get $\alpha_X \cdot x = 0$.

Conversely, $\alpha_X \cdot x = 0$ implies that $\sigma^*(\alpha_X) \cdot \sigma^*(x) = \sigma^*(\alpha_X \cdot x) = 0$. But we have just shown that $\sigma^*(\alpha_X) = \alpha_{G \times X}$ is a unit in $K_0^G(G \times X)$. So we must have $\sigma^*(x) = 0$. Composing this with $\tau_{G \times X}^G$ and applying Lemma 8.4, we conclude that $\bar{\sigma}^* \circ \tau_X^G(x) = 0$. This in turn gives $\bar{\sigma}_* \circ \bar{\sigma}^* \circ \tau_X^G(x) = 0$ and hence $|G|\tau_X^G(x) = 0$. \square

Corollary 8.6. *Let G be a finite group and let $X \in \mathcal{V}_G$. Then*

(i) *The maps*

$$\widetilde{G_i^G(X)} \rightarrow \widehat{G_i^G(X)} \rightarrow \widehat{G_i^G(X)}_{\mathfrak{m}_X} \leftarrow G_i^G(X)_{\mathfrak{m}_X}$$

are all isomorphisms.

(ii) *The Riemann-Roch map τ_X^G induces isomorphism*

$$\widetilde{G_i^G(X)} \xrightarrow[\cong]{\tau_X^G} CH_G^*(X, i).$$

(iii) *If X is smooth, then the Chern character map in Proposition 7.1 induces an isomorphism*

$$\widetilde{K_i^G(X)} \xrightarrow[\cong]{ch_X^G} CH_G^*(X, i).$$

Proof. First we note that α_X is a unit in $K_0^G(X)_{\mathfrak{m}_X}$ and so it acts as a unit on $G_i^G(X)_{\mathfrak{m}_X}$. It then follows from Corollary 7.6 and Theorem 8.5 that τ_X^G induces an isomorphism

$$(8.7) \quad G_i^G(X)_{\mathfrak{m}_X} \xrightarrow{\cong} CH_G^*(X, i).$$

To prove the isomorphism of the second and the third arrows in (i), it suffices to show that the \mathfrak{m}_X -adic filtration on $G_i^G(X)$ is nilpotent. But we have seen in the proof of Proposition 7.2 that $\overline{ch}_X^G(\mathfrak{m}_X^n) \subset \prod_{j \geq n} A_G^j(X)$ for every $n \geq 1$.

We conclude from Corollary 8.3 that $\mathfrak{m}_X^n CH_G^*(X, i) = 0$ for $n \gg 0$ and hence from 8.7, we get $\mathfrak{m}_X^n G_i^G(X)_{\mathfrak{m}_X} = 0$ for $n \gg 0$. To prove the isomorphism for the first arrow in (i), we note from [37, Theorem 1] that $R(G) = I_G \times (R(G)/I_G)$, where I_G is a finite product of fields. This implies that $I_G^n = I_G$ for all n and hence $\widehat{G_i^G(X)} \cong G_i^G(X)/I_G \cong \widehat{G_i^G(X)}$. The part (ii) now follows directly from the part (i) and 8.7.

If X is smooth, then Corollary 8.3 implies that there is a good pair (V, U) for the G -action such that $CH_G^*(X, \cdot) = CH^*(X_G, \cdot)$. Now we see from the construction of the non-equivariant Riemann-Roch map in the proof of Lemma 8.4, and its equivariant construction in 7.6 that

$$(8.8) \quad \tau_X^G = \frac{Td(X_G)}{Td(E_V)} ch_X^G.$$

Now the isomorphism of \widetilde{ch}_X^G follows from (ii) and the fact that the Todd classes are invertible elements in $CH^*(X_G, 0)$. \square

9. RIEMANN-ROCH ISOMORPHISM FOR ACTION WITH FINITE STABILIZERS

As was mentioned in the beginning, our approach to the proof of Theorem 1.2 is to eventually reduce the problem to the case when the underlying group acts either with finite stabilizers or with a constant dimension of stabilizers. In the case of action with finite stabilizers, we shall in fact prove the following stronger version of Theorem 1.2, where we do not assume that the given variety is smooth. The equivariant K -theory of smooth varieties for actions with finite stabilizers was studied before in [32] and [33]. The main result of [33] is a decomposition of the equivariant K -theory from which one can deduce the Riemann-Roch isomorphism. Our approach in this case is focussed more on directly constructing a Riemann-Roch isomorphism between the equivariant K -theory and higher Chow groups, without relying on the results of Toën and Vezzosi-Vistoli. Moreover, we prove this in a form which will be most suitable for us in the proof of our main result of this paper.

Theorem 9.1. *Let G be a linear algebraic group acting on a (possibly singular) variety X . If G acts with finite stabilizers, then the Riemann-Roch map of 7.5 gives rise to a commutative diagram*

$$(9.1) \quad \begin{array}{ccc} G_i^G(X) \otimes_{R(G)} \widehat{R(G)} & \xrightarrow{\widetilde{\tau}_X^G} & CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)} \\ u_X^G \downarrow & & \downarrow \overline{u}_X^G \\ \widehat{G_i^G(X)} & \xrightarrow{\tau_X^G} & \widehat{CH_G^*(X, i)}, \end{array}$$

where all the maps are isomorphisms. If G is a diagonalizable group, then the horizontal maps are isomorphisms and the vertical maps injective, even when it acts on X with a fixed dimension of stabilizers.

In this section, we study the Chern character and the Riemann-Roch maps for the torus action with finite stabilizers, and prove Theorem 9.1 in this special case. We begin with the following general result.

Proposition 9.2. *Let G be a linear algebraic group acting on a variety X with finite stabilizers. Then for any $i \geq 0$, one has $CH_G^j(X, i) = 0$ for $j \gg 0$. In particular, the natural maps*

$$CH_G^*(X, i) \rightarrow \widehat{CH_G^*(X, i)} \rightarrow \widehat{CH_G^*(X, i)} \rightarrow \overline{CH_G^*(X, i)}$$

are all isomorphisms.

Proof. We only need to prove the first assertion. The remaining part then follows exactly in the same way as the proof of Corollary 8.2. We embed G as a closed subgroup of GL_n for some n . Then GL_n acts naturally on the quotient $GL_n \overset{G}{\times} X$, and this action is with finite stabilizers if and only if G -action on X is with finite stabilizers (cf. [6, Section 5]). By Corollary 3.2, we can assume that G is the general linear group and hence a connected and split reductive group. We can now use Corollary 3.9 to reduce to the case when G is a split torus T .

Now, by Thomason's generic slice Theorem (cf. [31, Proposition 4.10]), there exists a non-empty T -invariant open subset $U \subset X$ and a diagonalizable subgroup $T_1 \subset T$ with quotient $T/T_1 = T_2$ such that T acts on U via T_2 , which in turn acts freely on U such that there is a T -equivariant isomorphism

$$U \xrightarrow{\cong} U/T \times T_2 \cong U/T \overset{T_1}{\times} T.$$

Since T acts on U with finite stabilizers, T_1 must be a finite diagonalizable group. Now we apply the Morita isomorphism of Corollary 3.2 to get

$$CH_T^*(U, i) \cong CH_T^* \left(U/T \overset{T_1}{\times} T, i \right) \cong CH_{T_1}^*(U/T, i).$$

Since T_1 acts trivially on U/T , we now apply Theorem 3.10 to get

$$(9.2) \quad CH_T^*(U, i) \cong CH^*(U/T, i) \otimes_{\mathbb{Q}} S(T_1).$$

Since $CH^j(U/T, i) = 0$ for $j \gg 0$ and since T_1 is a finite group, we must have $CH_T^j(U, i) = 0$ for $j \gg 0$. Now we apply the Noetherian induction and the localization sequence (cf. Proposition 2.2) to conclude that $CH_T^j(X, i) = 0$ for $j \gg 0$. \square

Lemma 9.3. *Let G and X be as in Proposition 9.2. Then the natural map*

$$G_i^G(X)_{I_G} \rightarrow \widehat{G_i^G(X)}$$

is an isomorphism.

Proof. This is a straightforward generalization of [6, Proposition 5.1] to higher K -theory and we only give the main steps. We only need to show that

$$I_G^n G_i^G(X)_{I_G} = 0 \quad \text{for } n \gg 0.$$

By embedding G into GL_n and using the Morita equivalence (cf. [27, Theorem 1.10])

$$(9.3) \quad G_i^{GL_n} \left(GL_n \times^G X \right) \xrightarrow{\cong} G_i^G(X),$$

we can reduce to the $G = GL_n$ case. In this case, we use the Merkurjev's isomorphism (cf. [22, Proposition 8.1])

$$G_i^G(X) \otimes_{R(G)} R(T) \xrightarrow{\cong} G_i^T(X)$$

to reduce to the case of $G = T$ a torus.

Now, we can use Thomason's generic slice Theorem as above and then [29, Lemma 5.6] to get a non-empty T -invariant open subset $U \subset X$ and a finite subgroup $T_1 \subset T$ such that there is a T -equivariant isomorphism $U \xrightarrow{\cong} U/T \times^{T_1} T$ and

$$G_i^T(U) \xrightarrow{\cong} G_i(U/T) \otimes_{\mathbb{Q}} R(T_1).$$

Since T_1 is finite, we have $I_{T_1} R(T_1) = 0$ by [37, Theorem 1], and hence $I_T G_i^T(U) = 0$. Now the corresponding result for $G_i^T(X)$ follows from the Noetherian induction and the localization sequence in the equivariant K -theory (cf. [27, Theorem 1.5]). \square

Theorem 9.4. *Let G be a diagonalizable group acting on a variety X with finite stabilizers. Then τ_X^G induces the following commutative diagram where all maps are isomorphisms.*

$$(9.4) \quad \begin{array}{ccc} \widetilde{G_i^G(X)} & \xrightarrow{\widetilde{\tau_X^G}} & \widetilde{CH_G^*(X, i)} \\ u_X^G \downarrow & & \downarrow \overline{u_X^G} \\ \widetilde{G_i^G(X)} & \xrightarrow{\widetilde{\tau_X^G}} & \widetilde{CH_G^*(X, i)} \end{array}$$

If X is smooth, then the Chern character map of Proposition 7.1 induces an isomorphism

$$\widetilde{K_i^G(X)} \xrightarrow{\widetilde{ch_X^G}} \widetilde{CH_G^*(X, i)}.$$

Proof. We see from Proposition 9.2 that the map $\widetilde{G_i^G(X)} \xrightarrow{\widetilde{\tau_X^G}} \widetilde{CH_G^*(X, i)}$ actually lifts to a map $\widetilde{G_i^G(X)} \xrightarrow{\widetilde{\tau_X^G}} \widetilde{CH_G^*(X, i)}$. The proposition also shows that the map $\overline{u_X^G}$ is an isomorphism. This automatically gives a natural map $\widetilde{\tau_X^G}$ as in the above diagram, which is functorial in X and which makes the diagram commute.

As in the proof of Lemma 9.3, we can choose a non-empty G -invariant open subset $U \subset X$ and a finite subgroup $T_1 \subset G$ such that $G_i^G(U) \xrightarrow{\cong} G_i(U/G) \otimes_{\mathbb{Q}} R(T_1)$, which in particular gives

$$G_i^G(U) \otimes_{R(G)} \widehat{R(G)} \xrightarrow{\cong} G_i(U/G) \otimes_{\mathbb{Q}} \left(R(T_1) \otimes_{R(G)} \widehat{R(G)} \right).$$

Since the natural map $R(G) \rightarrow R(T_1)$ is finite, we have

$$R(T_1) \otimes_{R(G)} \widehat{R(G)} \xrightarrow{\cong} \widehat{R(T_1)}_G \xrightarrow{\cong} \widehat{R(T_1)},$$

where the last equality follows from [6, Corollary 6.1]. Thus we get an isomorphism

$$(9.5) \quad G_i^G(U) \otimes_{R(G)} \widehat{R(G)} \xrightarrow{\cong} G_i(U/G) \otimes_{\mathbb{Q}} \widehat{R(T_1)}.$$

Similarly, we use 9.2 and [6, Theorem 6.1 (b)] to get an isomorphism

$$(9.6) \quad CH_G^*(U, i) \otimes_{S(G)} \widehat{S(G)} \xrightarrow{\cong} CH^*(U/G, i) \otimes_{\mathbb{Q}} \widehat{S(T_1)}$$

and a commutative diagram

$$(9.7) \quad \begin{array}{ccc} G_i^G(U) \otimes_{R(G)} \widehat{R(G)} & \xrightarrow{\cong} & G_i(U/G) \otimes_{\mathbb{Q}} \widehat{R(T_1)} \\ \tau_U^G \downarrow & & \downarrow \tau_{U/G} \otimes \tau_k^G \\ CH_G^*(U, i) \otimes_{S(G)} \widehat{S(G)} & \xrightarrow{\cong} & CH^*(U/G, i) \otimes_{\mathbb{Q}} \widehat{S(T_1)}. \end{array}$$

The map $\tau_{U/G}$ is an isomorphism by the non-equivariant Riemann-Roch (cf. [2, Theorem 9.1]) and τ_k^G is an isomorphism by Theorem 1.1. In particular, the right vertical map is an isomorphism and hence so is the left vertical map.

To prove the isomorphism of $\tilde{\tau}_X^G$, we let $Y = X - U$ and consider the diagram of localization sequences

$$\begin{array}{ccccccccc} \widetilde{G_{i+1}^G(U)} & \longrightarrow & \widetilde{G_i^G(Y)} & \longrightarrow & \widetilde{G_i^G(X)} & \longrightarrow & \widetilde{G_i^G(U)} & \longrightarrow & \widetilde{G_{i-1}^G(Y)} \\ \downarrow \tilde{\tau}_U^G & & \downarrow \tilde{\tau}_Y^G & & \downarrow \tilde{\tau}_X^G & & \downarrow \tilde{\tau}_U^G & & \downarrow \tilde{\tau}_Y^G \\ CH_G^*(U, i+1) & \rightarrow & CH_G^*(Y, i) & \rightarrow & CH_G^*(X, i) & \rightarrow & CH_G^*(U, i) & \rightarrow & CH_G^*(Y, i-1) \end{array}$$

which commutes by Theorem 7.5. The rows are exact since $\widehat{R(G)}$ and $\widehat{S(G)}$ are flat over $R(G)$ and $S(G)$ respectively. The map $\tilde{\tau}_Y^G$ is an isomorphism by the Noetherian induction, and we have shown above that $\tilde{\tau}_U^G$ is an isomorphism. We conclude from 5-lemma that $\tilde{\tau}_X^G$ is also an isomorphism.

To show that u_X^G is an isomorphism, the natural maps $G_i^G(X)_{I_G} \rightarrow \widetilde{G_i^G(X)} \rightarrow \widehat{G_i^G(X)}$ and Lemma 9.3 imply that u_X^G is surjective. It is injective because we have just shown that $\tilde{\tau}_X^G \circ u_X^G = \tilde{u}_X^G \circ \tilde{\tau}_X^G$ is an isomorphism. Finally, the isomorphism of $\tilde{\tau}_X^G$ now follows since all other maps in 9.4 are isomorphisms.

If X is smooth, we can use Proposition 9.2 to choose a good pair (V, U) for the G -action such that $CH_G^*(X, i) \cong CH^*(X_G, i)$. The rest of the argument is exactly the same as the proof of the corresponding result for finite group actions in Corollary 8.6. \square

10. PROOFS OF THEOREM 1.2 FOR DIAGONALIZABLE GROUPS

In this section, we prove Theorem 1.2 for the action of diagonalizable groups on smooth varieties. We first deal with the case when a diagonalizable group G acts on a variety (possibly singular) X such that the stabilizers have a constant dimension.

Proposition 10.1. *Let G be a split diagonalizable group and let $X \in \mathcal{V}_G$ be such that the stabilizers of all points of X have a constant dimension. Then Theorem 1.2 and Theorem 9.1 hold.*

Proof. Using Proposition 5.5 and Remark 5.6, we can assume that there exists a subtorus $T \subset G$ such that T acts trivially on X and $H = G/T$ acts on X with finite stabilizers. We have then

$$\begin{aligned}
 \widetilde{G_i^G(X)} &\cong G_i^G(X) \otimes_{R(G)} \widehat{R(G)} \\
 &\cong (G_i^H(X) \otimes_{\mathbb{Q}} R(T)) \otimes_{R(G)} \widehat{R(G)} \\
 &\cong G_i^H(X) \otimes_{R(H)} \left((R(H) \otimes_{\mathbb{Q}} R(T)) \otimes_{R(G)} (\widehat{R(H)} \otimes_{\mathbb{Q}} \widehat{R(T)}) \right) \\
 (10.1) \quad &\cong G_i^H(X) \otimes_{R(H)} \left(R(G) \otimes_{R(G)} (\widehat{R(H)} \otimes_{\mathbb{Q}} \widehat{R(T)}) \right) \\
 &\cong (G_i^H(X) \otimes_{R(H)} \widehat{R(H)}) \otimes_{\mathbb{Q}} \widehat{R(T)} \\
 &\cong \widetilde{G_i^H(X)} \otimes_{\mathbb{Q}} \widehat{R(T)},
 \end{aligned}$$

where the second isomorphism and the isomorphism $R(G) \cong R(T) \otimes_{\mathbb{Q}} R(H)$ follow from [29, Lemma 5.6]. We can similarly use Theorem 3.10 to get

$$(10.2) \quad \widetilde{CH_G^*(X, i)} \xrightarrow{\cong} \widetilde{CH_H^*(X, i)} \otimes_{\mathbb{Q}} \widehat{S(T)}.$$

First we assume $X \in \mathcal{V}_G^S$ and prove Theorem 1.2. Since the construction of the equivariant Chern character (cf. Proposition 7.1) at the individual factors of $\widetilde{CH_G^*(X, i)}$ is given by the non-equivariant Chern character on a mixed quotient, we have a commutative diagram

$$\begin{array}{ccccc}
 \widetilde{K_i^H(X)} \otimes_{\mathbb{Q}} \widehat{R(T)} & \xrightarrow{\cong} & \widetilde{K_i^G(X)} & \xrightarrow{u_X^G} & \widehat{K_i^G(X)} \\
 \widetilde{ch_X^H} \otimes \widetilde{ch_k^T} \downarrow & & \vdots \downarrow & & \downarrow \widehat{ch_X^G} \\
 \widetilde{CH_H^*(X, i)} \otimes_{\mathbb{Q}} \widehat{S(T)} & \xrightarrow{\cong} & \widetilde{CH_H^*(X, i)} & \xrightarrow{\overline{u}_X^G} & \overline{CH_H^*(X, i)},
 \end{array}
 \quad (10.3)$$

where the map \overline{u}_X^G is injective by Corollary 4.8. The left vertical map is an isomorphism by Theorem 9.4 and Theorem 1.1. The first arrow in both the rows are isomorphisms by 10.1 and 10.2. This automatically induces the middle vertical

arrow \widetilde{ch}_X^G , which is an isomorphism and makes both the squares commute. This also implies that u_X^G is injective. This proves Theorem 1.2. The proof of Theorem 9.1 follows exactly the same way by replacing the Chern character by the Riemann-Roch map, since Theorem 9.4 holds even when X is singular. \square

Theorem 10.2. *Let G be diagonalizable group and let $X \in \mathcal{V}_G^S$. Then the Chern character map ch_X^G induces an isomorphism*

$$\widetilde{K_i^G(X)} \xrightarrow{\widetilde{ch}_X^G} \widetilde{CH_G^*(X, i)}$$

such that the diagram

$$(10.4) \quad \begin{array}{ccc} K_i^G(X) \otimes_{R(G)} \widehat{R(G)} & \xrightarrow{\widetilde{ch}_X^G} & CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)} \\ u_X^G \downarrow & & \downarrow \overline{u}_X^G \\ \widehat{K_i^G(X)} & \xrightarrow{\widehat{ch}_X^G} & \overline{CH_H^*(X, i)} \end{array}$$

commutes.

Proof. In the rest of this section, we shall use the terms and the notations of Sections 5 and 6 freely without explaining them again. Let $s \geq 0$ be the largest integer such that $X_s \neq \emptyset$. We prove the theorem by induction on s . If $s = 0$, then G acts on X with finite stabilizers in which case the result is already proved in Theorem 9.4. So we assume that $s \geq 1$. Let $X_s \xrightarrow{f_s} X$ and $X_{<s} \xrightarrow{g_s} X$ be the inclusions of closed and open sets as before. Since G acts on X_s with constant dimension of stabilizers, the result holds for X_s by Proposition 10.1. This also holds for $X_{<s}$ and N_s^0 by induction on s (since $N_s^0 = (N_s)_{<s}$). Thus we have the maps $\widetilde{ch}_{X_{<s}}^G$ such that $\widehat{ch}_{X_{<s}}^G \circ u_{X_{<s}}^G = \overline{u}_{X_{<s}}^G \circ \widetilde{ch}_{X_{<s}}^G$ and similarly for X_s and N_s^0 . We now consider the following diagram.

$$(10.5) \quad \begin{array}{ccccc} \widetilde{K_i^G(X)} & \xrightarrow{(g_s^*, f_s^*)} & \widetilde{K_i^G(X_{<s})} \oplus \widetilde{K_i^G(X_s)} & \xrightarrow{Sp_{X,s}^{<s-1} - \eta_{s, \leq s-1}^*} & \widetilde{K_i^G(N_s^0)} \\ \downarrow \text{curved} & & \downarrow \widetilde{ch}_{X_{<s}}^G \oplus \widetilde{ch}_{X_s}^G & & \downarrow \widetilde{ch}_{N_s^0}^G \\ CH_G^*(X, i) & \xrightarrow{(\overline{g}_s^*, \overline{f}_s^*)} & CH_G^*(X_{<s}, i) \oplus CH_G^*(X_s, i) & \xrightarrow{\overline{Sp}_{X,s}^{<s-1} - \overline{\eta}_{s, \leq s-1}^*} & CH_G^*(N_s^0, i) \\ \downarrow \overline{u}_X^G & & \downarrow \overline{u}_{X_{<s}}^G \oplus \overline{u}_{X_s}^G & & \downarrow \overline{u}_{N_s^0}^G \\ \overline{CH_G^*(X, i)} & \xrightarrow{(\overline{g}_s^*, \overline{f}_s^*)} & \overline{CH_G^*(X_{<s}, i)} \oplus \overline{CH_G^*(X_s, i)} & \xrightarrow{\overline{Sp}_{X,s}^{<s-1} - \overline{\eta}_{s, \leq s-1}^*} & \overline{CH_G^*(N_s^0, i)}, \end{array}$$

where the curved downward arrow on the left is $\widehat{ch}_X^G \circ u_X^G$. The composite square on the left commutes by the contravariance of the Chern character. The lower right square clearly commutes. We show that the upper right square commutes.

For this, it suffices to show separately that

$$(10.6) \quad (i) \quad \widetilde{ch}_{N_s^0}^G \circ \eta_{s, \leq s-1}^* = \overline{\eta}_{s, \leq s-1}^* \circ \widetilde{ch}_{X_s}^G \quad \text{and} \quad (ii) \quad \widetilde{ch}_{N_s^0}^G \circ Sp_{\overline{X}, s}^{\leq s-1} = \overline{Sp}_{\overline{X}, s}^{\leq s-1} \circ \widetilde{ch}_{X_{< s}}^G.$$

But $\eta_{s, \leq s-1}^*$ is the composite of pull-backs

$$\widetilde{K}_i^G(X_s) \xrightarrow{\overline{f}_{s, \infty}^*} \widetilde{K}_i^G(N_s) \rightarrow \widetilde{K}_i^G(N_s^0),$$

where the last map is the pull-back to an open subset. The same factorization holds for the higher Chow groups as well. Hence (i) follows directly from the contravariance of the Chern character. To prove (ii), we consider the diagram

$$\begin{array}{ccccc} \widetilde{K}_i^G(M_{s, \leq s-1}) & \xrightarrow{j_{s, \leq s-1}^*} & \widetilde{K}_i^G(X_{< s}) & \xrightarrow{Sp_{\overline{X}, s}^{\leq s-1}} & \widetilde{K}_i^G(N_s^0) \\ \widetilde{ch}_{M_{s, \leq s-1}}^G \downarrow & & \downarrow \widetilde{ch}_{X_{< s}}^G & & \downarrow \widetilde{ch}_{N_s^0}^G \\ CH_G^*(M_{s, \leq s-1}, i) & \xrightarrow{\overline{j}_{s, \leq s-1}^*} & CH_G^*(X_{< s}, i) & \xrightarrow{\overline{Sp}_{\overline{X}, s}^{\leq s-1}} & CH_G^*(N_s^0, i). \end{array}$$

The left square is a pull-back diagram and hence commutes. The composite top horizontal arrow is just the pull-back $i_{s, \leq s-1}^*$ and the composite bottom horizontal arrow is $\overline{i}_{s, \leq s-1}^*$ (cf. Diagram 5.9). Hence the composite square commutes. The map $j_{s, \leq s-1}^*$ is surjective from the K -rigidity (cf. [34, Proposition 4.3, Proposition 4.6]). It follows that the right square also commutes, which proves 10.6. Hence we have shown that all the completed squares in the diagram 10.5 commute.

Now, the top row in the diagram 10.5 is exact with the first map injective by [34, Proposition 4.7] and the fact that $\widetilde{R}(T)$ is flat over $R(T)$. The same conclusion holds for the middle row by Proposition 6.1. The bottom row is exact with the first map injective by Proposition 6.1 and Proposition 4.1. It follows from a diagram chase that there exists a unique map $\widetilde{ch}_X^G : \widetilde{K}_i^G(X) \rightarrow CH_G^*(X, i)$ such that the upper left square commutes. The injectivity of $(\overline{g}_s^*, \overline{f}_s^*)$ now shows that $\widehat{ch}_X^G \circ u_X^G = \overline{u}_X^G \circ \widetilde{ch}_X^G$. This proves the theorem. \square

11. REDUCTION STEPS FOR ARBITRARY GROUPS

In this section, we establish some reduction steps for deducing the proofs of our main results for an arbitrary group from the case of diagonalizable groups. We first do this for $G = GL_n$.

Proposition 11.1. *Let G be a linear algebraic group and let $X \in \mathcal{V}_G$. Suppose that Theorem 1.2 (resp. Theorem 9.1) holds when G is a diagonalizable group. Then Theorem 1.2 (resp. Theorem 9.1) also holds when $G = GL_n$.*

Proof. We first prove the assertion for Theorem 1.2. The other case is similar and we shall indicate the specific changes. So let T be a split maximal torus of $G = GL_n$ and W the Weyl group. Then by Corollary 3.9, W acts on $CH_T^*(X, \cdot)$ as graded automorphisms and $CH_G^j(X, i) = (CH_T^j(X, i))^W$ for every $i, j \geq 0$. In particular,

W acts on $\overline{CH_T^*(X, i)}$ and $\overline{CH_G^*(X, i)} = \left(\overline{CH_T^*(X, i)}\right)^W$. Since W acts through the pull-backs, we see that it acts on $\widehat{S(T)}$ as $\widehat{S(G)}$ -algebra automorphisms and on $\overline{CH_T^*(X, i)}$ as $\widehat{S(G)}$ -linear maps. We also have the trace maps (cf. 8.2)

$$\overline{CH_G^*(X, i)} \hookrightarrow \overline{CH_T^*(X, i)} \xrightarrow{tr} \overline{CH_G^*(X, i)}$$

such that the composite is identity. Letting W act on $CH_T^*(X, i) \otimes_{S(G)} \widehat{S(T)}$ diagonally, we get natural maps

$$CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)} \rightarrow CH_T^*(X, i) \otimes_{S(G)} \widehat{S(T)} \xrightarrow{tr} CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)}$$

such that the composite is identity. We note here that since $S(T)$ and $S(G)$ are polynomial algebras, the J_T -adic and the graded filtrations on $S(T)$ coincide, and similarly for $S(G)$. In particular, the maps $\widehat{S(T)} \rightarrow \overline{CH_T^*(k, 0)}$ and $\widehat{S(G)} \rightarrow \overline{CH_G^*(k, 0)}$ are isomorphisms.

Since W preserves the graded filtration of $CH_T^*(X, i)$, it preserves the augmentation ideal J_T of $S(T)$. Moreover, as W acts as an $S(G)$ -algebra on $S(T)$, we see that W preserves all powers of J_G and hence the J_G -adic filtration of $CH_T^*(X, i)$. Since W clearly preserves the J_G -adic filtration on $S(T)$ and since this filtration induces the same topology on $S(T)$ as the J_T -adic filtration (cf. [6, Theorem 6.1]), we see that the maps

$$CH_T^*(X, i) \otimes_{S(G)} S(T) \rightarrow CH_T^*(X, i) \otimes_{S(G)} \frac{S(T)}{J_G^j S(T)} \rightarrow \frac{CH_T^*(X, i)}{J_T^j CH_T^*(X, i)} \rightarrow \frac{CH_T^*(X, i)}{CH_T^{\geq j}(X, i)}$$

are all W -equivariant. We conclude that the composite map

$$(11.2) \quad \begin{array}{ccc} CH_T^*(X, i) \otimes_{S(G)} \widehat{S(T)} & \xrightarrow{w_X^G} & \overline{CH_T^*(X, i)} \\ & \searrow & \nearrow \\ & CH_T^*(X, i) \otimes_{S(T)} \widehat{S(T)} & \end{array}$$

is W -equivariant, where the slanted downward arrow is the natural surjection and the upward arrow is injective by Corollary 4.8. We now have the following diagram.

$$\begin{array}{ccccc} CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)} & \hookrightarrow & CH_T^*(X, i) \otimes_{S(G)} \widehat{S(T)} & \xrightarrow{tr} & CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)} \\ \bar{w}_X^G \downarrow & & w_X^G \downarrow & & \downarrow \bar{w}_X^G \\ \overline{CH_G^*(X, i)} & \hookrightarrow & \overline{CH_T^*(X, i)} & \xrightarrow{tr} & \overline{CH_G^*(X, i)} \end{array}$$

The left square clearly commutes since W acts trivially on the terms in the left column. Since w_X^G is W -equivariant and \bar{w}_X^G is its restriction on $CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)}$,

the right square also commutes. Since both the composite horizontal arrows are identity, we get the natural maps

$$(11.3) \quad \frac{\overline{CH_G^*(X, i)}}{CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)}} \hookrightarrow \frac{\overline{CH_T^*(X, i)}}{CH_T^*(X, i) \otimes_{S(G)} \widehat{S(T)}} \xrightarrow{tr} \frac{\overline{CH_G^*(X, i)}}{CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)}}$$

such that the composite is identity. Moreover, the middle term is same as $\frac{\overline{CH_T^*(X, i)}}{\overline{CH_T^*(X, i)}}$ by 11.2. We now consider the following diagram.

$$(11.4) \quad \begin{array}{ccc} \widetilde{K_i^G(X)} & \cdots \cdots \cdots \rightarrow & \widetilde{CH_G^*(X, i)} \\ \downarrow u_X^G & & \downarrow \bar{u}_X^G \\ \widetilde{K_i^G(X)} & \xrightarrow{\widetilde{ch}_X^G} & \overline{CH_G^*(X, i)} \\ & \searrow & \downarrow \\ & & \widetilde{K_i^T(X)} \xrightarrow{\widetilde{ch}_X^T} \widetilde{CH_T^*(X, i)} \\ & \searrow & \downarrow \bar{u}_X^T \\ & & \widetilde{K_i^T(X)} \xrightarrow{\widetilde{ch}_X^T} \overline{CH_T^*(X, i)} \end{array}$$

All the completed faces of the above diagram commute and the map \widetilde{ch}_X^T is an isomorphism by our assumption about the torus action. We want to show that the dotted arrow can be completed by the map \widetilde{ch}_X^G such that the back face (and hence all faces) of the cube commutes. Using 11.3 and a diagram chase, we see that it suffices to show that the image of the composite map

$$\widetilde{K_i^G(X)} \xrightarrow{\widetilde{ch}_X^G \circ u_X^G} \overline{CH_G^*(X, i)} \rightarrow \overline{CH_T^*(X, i)}$$

lies in $\overline{CH_T^*(X, i)}$. But this follows directly from the commutativity of all completed squares in the above diagram.

It remains now to show that \widetilde{ch}_X^G is an isomorphism. We first note that the completed arrows in the above diagram are $\widehat{R(G)}$ -linear via the isomorphism of the rings $\widehat{R(G)} \xrightarrow{\cong} \widehat{S(G)}$. In particular, \widetilde{ch}_X^G is $\widehat{R(G)}$ -linear. Moreover, $\widehat{R(T)} \cong \widehat{S(T)}$ is faithfully flat over $\widehat{R(G)} \cong \widehat{S(G)}$. Thus it suffices to show that $\widetilde{ch}_X^G \otimes_{\widehat{R(G)}} \widehat{R(T)}$

is an isomorphism. However, we have the following commutative diagram.
(11.5)

$$\begin{array}{ccc}
\widehat{K_i^G(X) \otimes_{\widehat{R(G)}} \widehat{R(T)}} & \xrightarrow{\widetilde{ch}_X^G \otimes \widetilde{ch}_k^G} & \widehat{CH_G^*(X, i) \otimes_{\widehat{S(G)}} \widehat{S(T)}} \\
\cong \downarrow & & \downarrow \cong \\
K_i^G(X) \otimes_{R(G)} \widehat{R(T)} & & CH_G^*(X, i) \otimes_{S(G)} \widehat{S(T)} \\
\cong \downarrow & & \downarrow \cong \\
(K_i^G(X) \otimes_{R(G)} R(T)) \otimes_{R(T)} \widehat{R(T)} & & (CH_G^*(X, i) \otimes_{S(G)} S(T)) \otimes_{S(T)} \widehat{S(T)} \\
\psi_T^G \downarrow & & \downarrow \phi_T^G \\
K_i^T(X) \otimes_{R(T)} \widehat{R(T)} & \xrightarrow{\widetilde{ch}_X^T \otimes \widetilde{ch}_k^T} & CH_T^*(X, i) \otimes_{S(T)} \widehat{S(T)}
\end{array}$$

The map ψ_T^G is an isomorphism by [22, Proposition 8] and ϕ_T^G is an isomorphism by Theorem 3.8. The bottom horizontal arrow is an isomorphism by our assumption. We conclude that the top horizontal arrow is also an isomorphism.

For the proof of Theorem 9.1 for $G = GL_n$, we first observe that 11.3 is general and it holds for any reductive group G with a split maximal torus T and any $X \in \mathcal{V}_G$ by Corollary 3.9. Hence we can consider the same diagram as 11.4 with K -groups replaced by G -groups and the Chern character replaced by the equivariant Riemann-Roch of Theorem 7.5 to get the desired map $\widetilde{\tau}_X^G$. Then all the maps in the diagram are $\widehat{R(G)}$ -linear by Theorem 7.5 (ii). Since G acts with finite stabilizers, so does T and hence $\widetilde{\tau}_X^T$ is an isomorphism by Proposition 10.1. Moreover, as [22, Proposition 8] also holds for the G -theory, the same argument as above shows that $\widetilde{\tau}_X^G$ is an isomorphism. \square

We need the following finiteness results for the maps of the equivariant Chow rings and the completions of the representation rings for generalizing Proposition 11.1 to any linear algebraic group. For compact Lie groups and complex algebraic groups, Lemma 11.3 follows from the stronger results of Segal (cf. [23]) and Edidin-Graham (cf. [8, Proposition 2.3]).

Lemma 11.2. *Let G be a linear algebraic group and let $H \subset G$ be a closed subgroup. Then the restriction map $S(G) \rightarrow S(H)$ is finite. In particular, $S(G)$ is Noetherian.*

Proof. We first assume that G is a diagonalizable group and $H \subset G$ is a subtorus. Then we have an isomorphism $R(G) \cong R(H) \otimes_{\mathbb{Q}} R(G/H)$ by [29, Lemma 5.6] and the natural map $R(G) \rightarrow R(H)$ is surjective and hence finite. In general, we embed G inside some GL_n to get the maps $R(GL_n) \rightarrow R(G) \rightarrow R(H)$. Hence we can assume that $G = GL_n$.

There is a finite extension l/k such that all the connected components of the algebraic group H_l are defined over l , the identity component H_l^0 is split and its unipotent radical $R_u(H_l^0)$ is also defined and split over l . Moreover, there is

maximal torus T of G_l such that $T \cap H_l = T'$ is a split maximal torus of H_l . We now consider the following commutative diagram.

$$(11.6) \quad \begin{array}{ccccc} S(G) & \xrightarrow{\cong} & S(G_l) & \longrightarrow & S(T) \\ \downarrow & & \downarrow & & \downarrow \\ S(H) & \longrightarrow & S(H_l) & \longrightarrow & S(T') \end{array}$$

We have shown above that the right vertical arrow is finite. The first horizontal arrow on the top is an isomorphism since $G = GL_n$ (cf [6, Section 3.2]), and the second horizontal arrow on the top is finite by Corollary 3.9. In particular, $S(T')$ is finite over $S(G)$. To see that the middle vertical arrow is finite, we can use Lemma 4.2 and Corollary 8.3 to assume that H_l is connected and split reductive with split maximal torus T' . In that case, it follows from Corollary 3.9 and the fact that $S(G)$ is Noetherian. Finally, we conclude from Lemma 4.4 that $S(H)$ is an $S(G)$ -submodule of $S(H_l)$ and hence finite over $S(G)$. The claim about the Noetherian property follows from this finiteness result and the fact that $S(GL_n)$ is Noetherian. \square

Lemma 11.3. *Let G be a linear algebraic group and let $H \subset G$ be a closed subgroup. Then the restriction maps $R(H) \otimes_{R(G)} \widehat{R(G)} \xrightarrow{r_H^G} \widehat{R(H)}$ and $S(H) \otimes_{S(G)} \widehat{S(G)} \xrightarrow{r_H^G} \widehat{S(H)}$ are isomorphisms of \mathbb{Q} -algebras.*

Proof. Let $\widehat{R(H)}_G$ and $\widehat{S(H)}_G$ denote the I_G -adic completion of $R(H)$ and J_G -adic completion of $S(H)$ respectively. Then we have $\widehat{R(H)}_G \xrightarrow{\cong} \widehat{R(H)}$ and $\widehat{S(H)}_G \xrightarrow{\cong} \widehat{S(H)}$ by [6, Theorem 6.1]. The isomorphism of the map $S(H) \otimes_{S(G)} \widehat{S(G)} \xrightarrow{r_H^G} \widehat{S(H)}$ now follows directly from Lemma 11.2. We consider the following commutative diagram.

$$(11.7) \quad \begin{array}{ccccccc} R(G) & \longrightarrow & \widehat{R(G)} & \xlongequal{\quad} & \widehat{R(G)} & \xrightarrow{\widehat{ch}_k^G} & \widehat{S(G)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R(H) & \longrightarrow & R(H) \otimes_{R(G)} \widehat{R(G)} & \longrightarrow & \widehat{R(H)}_G & \cdots \cdots \cdots & \widehat{S(H)}_G \\ & & & & \downarrow \cong & & \downarrow \cong \\ & & & & \widehat{R(H)} & \xrightarrow{\widehat{ch}_k^H} & \widehat{S(H)} \end{array}$$

Since \widehat{ch}_k^G and \widehat{ch}_k^H are ring isomorphisms by Theorem 1.1, we conclude that there exists a unique isomorphism $\widehat{R(H)}_G \xrightarrow{\widehat{ch}_k^H} \widehat{S(H)}_G$ such that the above diagram commutes. We have shown that $\widehat{S(H)}_G \cong S(H) \otimes_{S(G)} \widehat{S(G)}$, and the latter is then

finite over $\widehat{S(G)}$. Now we conclude from the above diagram that $\widehat{R(H)}_G$ is finite over $\widehat{R(G)}$.

Using the indicated isomorphisms of rings in the above diagram, we need to show that the composite map $R(H) \otimes_{R(G)} \widehat{R(G)} \xrightarrow{\phi_H^G} \widehat{S(H)}_G$ in the middle row is an isomorphism, in order to prove the proposition. We have now the natural maps

$$R(H) \otimes_{R(G)} \widehat{R(G)} \rightarrow K_0^G(G/H) \otimes_{R(G)} \widehat{R(G)} \xrightarrow{\widetilde{ch}_{G/H}^G} CH_G^*(G/H, 0) \otimes_{S(G)} \widehat{S(G)}.$$

The first map is an isomorphism by the Morita equivalence [6, Proposition 3.2] and the second map is an isomorphism by Theorem 10.2 and Proposition 11.1. However, the last term is same as $CH_H^*(k, 0) \otimes_{S(G)} \widehat{S(G)} \cong S(H) \otimes_{S(G)} \widehat{S(G)}$ by [6, Proposition 3.2], and we have seen above that the latter term is same as $\widehat{S(H)}_G$. This proves the lemma. \square

Proposition 11.4. *Let G be a linear algebraic group and let $X \in \mathcal{V}_G$. Suppose that Theorem 1.2 (resp. Theorem 9.1) holds when G is a diagonalizable group. Then Theorem 1.2 (resp. Theorem 9.1) holds for any G .*

Proof. As in the proof of Proposition 11.1, we first prove the assertion for Theorem 1.2. The other case is similar and we shall indicate the specific changes. Choosing an embedding $G \hookrightarrow GL_n$ as a closed subgroup and using [27, Theorem 1.10] and Corollary 3.2 above, we have natural isomorphisms

$$G_i^{GL_n} \left(GL_n \overset{G}{\times} X \right) \xrightarrow{\cong} G_i^G(X) \quad \text{and}$$

$$CH_{GL_n}^* \left(GL_n \overset{G}{\times} X, i \right) \xrightarrow{\cong} CH_G^*(X, i).$$

Putting $\tilde{X} = GL_n \times^G X$ and replacing the G -groups by K -groups (as X is smooth), we get the following commutative diagram.

$$\begin{array}{ccc}
K_i^G(X) \otimes_{R(G)} \left(R(G) \otimes_{R(GL_n)} \widehat{R(GL_n)} \right) & \xrightarrow{\cong} & K_i^{GL_n}(\tilde{X}) \otimes_{R(GL_n)} \widehat{R(GL_n)} \\
\downarrow r_G^{GL_n} & & \downarrow \widetilde{ch}_{\tilde{X}}^{GL_n} \\
& & CH_{GL_n}^*(\tilde{X}, i) \otimes_{S(GL_n)} \widehat{S(GL_n)} \\
& & \downarrow \\
& & CH_G^*(X, i) \otimes_{S(GL_n)} \widehat{S(GL_n)} \\
& & \downarrow \cong \\
& & CH_G^*(X, i) \otimes_{S(G)} \left(S(G) \otimes_{S(GL_n)} \widehat{S(GL_n)} \right) \\
& & \downarrow r_G^{GL_n} \\
K_i^G(X) \otimes_{R(G)} \widehat{R(G)} & \xrightarrow{\widetilde{ch}_X^G} & CH_G^*(X, i) \otimes_{S(G)} \widehat{S(G)} \\
\downarrow u_X^G & & \downarrow \overline{u}_X^G \\
\widehat{K_i^G(X)} & \xrightarrow{\widetilde{ch}_X^G} & \overline{CH_G^*(X, i)}
\end{array}$$

The top horizontal map is the composite

$$\begin{aligned}
K_i^G(X) \otimes_{R(G)} \left(R(G) \otimes_{R(GL_n)} \widehat{R(GL_n)} \right) &\xrightarrow{\cong} K_i^G(X) \otimes_{R(GL_n)} \widehat{R(GL_n)} \\
&\xrightarrow{\cong} K_i^{GL_n}(\tilde{X}) \otimes_{R(GL_n)} \widehat{R(GL_n)}.
\end{aligned}$$

The second vertical arrow on the right is an isomorphism as mentioned above. The maps $r_G^{GL_n}$ on both sides are isomorphisms by Lemma 11.3, $\widetilde{ch}_{\tilde{X}}^{GL_n}$ is an isomorphism by Proposition 11.1 and \overline{u}_X^G is injective by Corollary 4.8. We conclude that the map $\widehat{ch}_X^G \circ u_X^G$ factors through a map \widetilde{ch}_X^G which is an isomorphism.

To deduce Theorem 9.1 from the case of diagonalizable groups, we use the same diagram as above for G -groups and the same changes as in the proof of the theorem for GL_n in Proposition 11.1, and observe that if G acts on X with finite stabilizers, then so does GL_n on \tilde{X} . \square

12. PROOF OF THE MAIN RESULT AND CONSEQUENCES

Our strategy to prove Theorem 1.2 and Theorem 9.1 is to use the results of the previous section to reduce the proofs to the case of diagonalizable groups.

Proof of Theorem 1.2: The existence of \widetilde{ch}_X^G and its isomorphism follows from Theorem 10.2 and Proposition 11.4. We have moreover seen from the construction

of the equivariant Chern character in Proposition 7.1 that $\widehat{R(G)} \xrightarrow{\widehat{ch}_k^G} \widehat{S(G)}$ is an isomorphism of rings and \widehat{ch}_X^G is $\widehat{R(G)}$ -linear under this isomorphism. Taking the $\widehat{I_G}$ -completion of the top row in the diagram 1.3 under the isomorphism $\widehat{I_G} \cong \widehat{J_G}$, we see that the isomorphism \widehat{ch}_k^G induces the isomorphism of completions $\widehat{K_i^G(X)} \xrightarrow{\widehat{ch}_k^G} \widehat{CH_G^*(X, i)}$.

Finally, the map $\widehat{u_X^G}$ is injective by Corollary 4.8. Hence u_X^G is also injective. \square

Proof of Corollary 1.3: The ring homomorphism $\widehat{K_0^G(X)} \xrightarrow{\widehat{ch}_X^G} \widehat{CH_G^*(X, 0)}$ is an isomorphism by Theorem 1.1. Moreover, the map $\widehat{\tau_X^G}$ is a $\widehat{K_0^G(X)}$ -linear under the above isomorphism by Theorem 7.5. Since the Todd classes of the vector bundles are invertible elements of $\widehat{CH_G^*(X, 0)}$, we see from the construction of the Riemann-Roch map in Theorem 7.5 that there is an invertible element β_X in $\widehat{K_0^G(X)}$ such that $\widehat{\tau_X^G} = \beta_X \cdot \widehat{ch}_X^G$. The corollary now follows from Theorem 1.2. \square

Proof of Theorem 9.1: If a diagonalizable group acts with a fixed dimension of stabilizers, then this is proved in Proposition 10.1. If any linear algebraic group G acts on X with finite stabilizers, then $\widehat{\tau_X^G}$ is an isomorphism by the case of diagonalizable groups and Proposition 11.4. Since $\widehat{\tau_X^G}$ is an $\widehat{R(G)}$ -linear (cf. Theorem 7.5), the isomorphism of $\widehat{G_i^G(X)} \xrightarrow{\widehat{\tau_k^G}} \widehat{CH_G^*(X, i)}$ follows exactly in the same way as the isomorphism of \widehat{ch}_X^G in the proof of Theorem 1.2.

To prove the isomorphism of vertical arrows in the diagram 9.1, we first see that the map $\widehat{u_X^G}$ is an isomorphism by Proposition 9.2. This implies then that u_X^G is injective. On the other hand, the natural maps $G_i^G(X)_{I_G} \rightarrow \widetilde{G_i^G(X)} \xrightarrow{u_X^G} \widehat{G_i^G(X)}$ and Lemma 9.3 imply that u_X^G is surjective. \square

Proof of Theorem 1.4: For any $X \in \mathcal{V}_G$, we have the natural maps

$$G_i^G(X) \rightarrow G_i^G(X)_{I_G} \xrightarrow{\widehat{\tau_X^G}} \widetilde{CH_G^*(X, i)} \leftarrow CH_G^*(X, i),$$

where the last map is an isomorphism by Proposition 9.2 and $\widehat{\tau_X^G}$ is an isomorphism by Theorem 9.1. This gives the desired lifting of the Riemann-Roch map $G_i^G(X) \xrightarrow{\tau_X^G} CH_G^*(X, i)$ such that $\text{Ker}(\tau_X^G)$ has the desired property.

For the surjectivity of τ_X^G , we first prove the case when X is smooth. By [24, Theorem 6.1], there is a finite G -equivariant cover $f : X' \rightarrow X$ such that X' is normal and G acts freely on X' . In particular, X' is quasi-projective and the G -action on X' is linear (cf. [9, Proposition 7.1]). We claim that for any $i, j \geq 0$, the map $CH_G^j(X', i) \xrightarrow{f_*} CH_G^j(X, i)$ is surjective. To see this, choose a representation V of G and a G -invariant open subset U of V such that G acts freely on U and the codimension of $V - U$ is larger than j . Then the induced map $\tilde{f} : X'_G \rightarrow X_G$ is a

finite map of quasi-projective schemes (*loc. cit.*) such that X_G is smooth. Hence by [2, Theorem 4.1 and 5.8], there is a pull-back map $\tilde{f}^* : CH^j(X_G, i) \rightarrow CH^j(X'_G, i)$ such that $\tilde{f}_* \circ \tilde{f}^*$ is the multiplication by the degree of \tilde{f} , which proves the claim.

Now let $Y' = X'/G$ and consider the following Riemann-Roch diagram.

$$\begin{array}{ccc} G_i^G(X') & \xrightarrow{\tau_{X'}^G} & CH_G^*(X', i) \\ f_* \downarrow & & \downarrow \tilde{f}_* \\ G_i^G(X) & \xrightarrow{\tau_X^G} & CH_G^*(X, i) \end{array}$$

Note that like X'_G and X_G , Y' is also quasi-projective. Moreover, one has $G_i^G(X') \cong G^i(Y')$ and $CH_G^*(X', i) \cong CH^*(Y', i)$. In particular, we conclude from the Bloch's non-equivariant Riemann-Roch Theorem that the top horizontal arrow is an isomorphism. We have just shown above that the right vertical map is surjective. We conclude that the bottom horizontal map must be surjective.

If X is singular and the stack $[X/G]$ has a coarse moduli scheme, then by [20, Theorem 1], there a $X' \in \mathcal{V}_G$ and finite flat G -equivariant map $f : X' \rightarrow X$ such that G acts freely on X' . As f is finite and flat, we have maps $\tilde{f}^* : CH^j(X_G, i) \rightarrow CH^j(X'_G, i)$ and $\tilde{f}_* : CH^j(X'_G, i) \rightarrow CH^j(X_G, i)$ such that $\tilde{f}_* \circ \tilde{f}^*$ is the multiplication by the degree of \tilde{f} by the same reason. The same argument as in the smooth case above now proves that τ_X^G is surjective. \square

As another application of our Riemann-Roch isomorphisms to the equivariant K -theory, we can prove the following partial generalization of [22, Proposition 8] from simply connected to arbitrary reductive groups. For simply connected reductive groups, Merkurjev's result is stronger in that his result holds with the integral coefficients. But the main advantage of the following result is that it does not assume anything about G . It will turn out in the forthcoming sequel that it may not be necessary to complete the ring $R(T)$ in the corollary below.

Corollary 12.1. *Let G be a connected and split reductive group and let T be a split maximal torus of G . Then for any smooth variety X with G -action and for any $i \geq 0$, the natural map*

$$K_i^G(X) \otimes_{R(G)} \widehat{R(T)} \xrightarrow{\eta_T^G} K_i^T(X) \otimes_{R(T)} \widehat{R(T)}$$

is an isomorphism. For actions with finite stabilizers, η_T^G is an isomorphism even if X is not smooth.

Proof. We first note that there is an $\widehat{R(G)}$ -linear isomorphism

$$K_i^G(X) \otimes_{R(G)} \widehat{R(T)} \xrightarrow{\cong} \widetilde{K_i^G(X)} \otimes_{\widehat{R(G)}} \widehat{R(T)}.$$

Thus we need to show that the natural map $\widetilde{K_i^G(X)} \otimes_{\widehat{R(G)}} \widehat{R(T)} \rightarrow \widetilde{K_i^T(X)}$ is an isomorphism. For this, we consider the following commutative diagram.

$$\begin{array}{ccc} \widetilde{K_i^G(X)} \otimes_{\widehat{R(G)}} \widehat{R(T)} & \longrightarrow & \widetilde{K_i^T(X)} \\ \downarrow & & \downarrow \\ \widetilde{CH_G^*(X, i)} \otimes_{\widehat{S(G)}} \widehat{S(T)} & \longrightarrow & \widetilde{CH_T^*(X, i)} \end{array}$$

The vertical maps are isomorphisms by Theorem 1.2. So we need to show that the bottom horizontal map is an isomorphism. However, we have isomorphism

$$\begin{aligned} \widetilde{CH_G^*(X, i)} \otimes_{\widehat{S(G)}} \widehat{S(T)} &\cong CH_G^*(X, i) \otimes_{S(G)} \left(\widehat{S(G)} \otimes_{\widehat{S(G)}} \widehat{S(T)} \right) \\ &\cong (CH_G^*(X, i) \otimes_{S(G)} S(T)) \otimes_{S(T)} \widehat{S(T)}. \end{aligned}$$

But the last term is naturally isomorphic to $\widetilde{CH_T^*(X, i)} \otimes_{S(T)} \widehat{S(T)} = \widetilde{CH_T^*(X, i)}$ by Theorem 3.8. This proves the corollary when X is smooth. The same proof also applies when X is not smooth and G acts with finite stabilizers by Theorem 9.1 since our extra ingredient Theorem 3.8 holds for all $X \in \mathcal{V}_G$. \square

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